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CONSTRUCTION OF NESTED INCOMPLETE SPLIT-PLOT × SPLIT-BLOCK DESIGNS

Iwona Mejza, Katarzyna Ambroży

Department of Mathematical and Statistical Methods
Agricultural University of Poznań
Wojska Polskiego 28, 60-637 Poznań
e-mail:imejza@au.poznan.pl

Summary

Nested blocking structure of a design allows controlling several sources of local variation in an experiment. This fact we use in a designing and an analysing three factor experiments with some nested and crossed structure of units. The designs considered can be incomplete with respect to one, two or all the factors which levels are arranged in nested incomplete block designs, in particular in resolvable ones. Then resulting designs are called Nested Split-Plot × Split-Block (say NSPSB) designs. Performed randomization processes according to a stratification of the experimental units lead to eight strata in the analysis. One of them, called zero stratum, is connected with an estimation of an experimental mean only. In other strata analyses can be done. In the paper we present situations when comparisons among main effects and interaction effects of the factors are estimable in the different strata. The considerations are illustrated with examples of the construction of efficient equireplicate and nonequireplicate NSPSB designs.

Key words and phrases: general balance, multistratum experiments, nested designs, resolvable incomplete block designs, split- plot × split- block designs, stratum efficiency factors

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1. Introduction

In practice, particularly in agricultural field experiments situations appear often in which experimental blocks are grouped into some sets called “superblocks". Such nested blocking structure allows controlling several sources of local variation. In the paper we consider similar case with reference to a three-factor experiment set up in a split-plot × split-block (say SPSB) design (e.g. LeClerg et al., 1962).

In the ordinary SPSB design a randomization processes according to a stratification of the experimental units leads to seven strata: zero stratum (0), the inter-block stratum (1), the inter-row (within the block) stratum (2), the inter-column I (within the block) stratum (3), the inter-column II stratum (4) (within the column I), the inter-whole plot (within the block) stratum (5), and the inter-subplot (within the whole plot) stratum (6), (cf. Ambroży and Mejza I., 2003, 2004a, 2004b). The number of the strata will increase when we take into account the nested structure of the blocks. Then we have additional stratum more called the inter-superblock stratum and other strata apart (0), mentioned above, follow it. Then the resulting design called the nested split-plot × split-block (say, NSPSB) design is said to be incomplete with respect to the blocks but complete with respect to the superblocks. Incompleteness can be related to one factor only, two factors only or all the factors.

The aim of the paper is to present a randomization model, statistical properties and their consequences for an analysis of some three factor experiments set up in a NSPSB designs. A concept of resolvability in nested block designs described by Caliński and Kageyama (2000a, 2000b) has been adopted for the designs considered.

2. Material structure

Let us consider a three-factor experiment of NSPSB type in which the first factor, say *A*, has *s* levels A_1, A_2, …, A_s, the second factor, say *B*, has *t* levels B_1, B_2, …, B_t and the third factor, say *C*, has *w* levels C_1, C_2, …, C_w. Thus the number *v* (= *stw*) denotes the number of all treatment combinations in the experiment. To control several sources of local variation *experimental material* is assumed to be divided into *R superblocks* consisting of *b blocks*, each block has a row–column structure with k_1 ($\le s$) *rows* (horizontal strips) and k_2 ($\le t$) columns (vertical strips I) of the first order, shortly, *columns I*. So there are k_1k_2 intersection plots of the first order within each block, below called *whole plots*. Then each column I is split into k_3 ($\le w$) columns (vertical strips II) of the second order, shortly, *columns II*. So there are $k_1k_2k_3$ intersection plots of the second order within each block, below called *small plots*. Here the rows correspond to the levels of the factor *A*, termed also as *row treatments*, the columns I correspond to the levels of the factor *B*, called also *column I treatments*, and the columns II are to accommodate the levels of the factor *C* termed as *column II treatments*. The arrangement of the factors in the NSPSB design is very important from the statistical point of view. It has an affect on a precision of estimation of treatment combination comparisons.

3. Linear model and its analysis

Considered here randomization model of observations has a form and statistical properties strictly connected with performed randomization processes in the experiment. It is based on an extension of that developed by Ambroży and Mejza (2003, 2004a) for the SPSB design. Used here a randomization scheme consists of five randomization steps performed independently, i.e. by randomly permuting the superblocks within a total area, by randomly permuting blocks within the superblocks, then the rows within the blocks, the columns I within the blocks and the columns II within the columns I in blocks. Further, assuming the usual unit-treatment additivity and uncorrelation of the technical errors, with zero expectation and a constant variance σ_e^2, the model of observations can be written as

$$\mathbf{y} = \boldsymbol{\Delta}'\boldsymbol{\tau} + \sum_{f=1}^{7} \mathbf{D}'_f \boldsymbol{\eta}_f + \mathbf{e}, \tag{3.1}$$

where $\mathbf{y}$ is an $n \times 1$ vector of lexicographically ordered observations, $n = Rbk_1k_2k_3$, $\boldsymbol{\Delta}'$ $(n \times v)$ is a known design matrix for v treatment combinations, $\mathbf{D}'_f$ (f =1,2,...,7) are design matrices for the superblocks, the blocks (within the superblocks), the rows (within the blocks), the columns I (within the blocks), the column II (within the columns I), the whole plots (within the blocks) and the subplots (within the whole plots) respectively, $\boldsymbol{\tau}$ $(v \times 1)$ is the vector of fixed treatment combination effects, $\boldsymbol{\eta}_f$ (f = 1,2,...,7) are random effect vectors of the superblocks, the blocks, the rows, the columns I, the columns II, the whole plots, the subplots and the technical errors, respectively.

Then under accepted assumptions we can write the first two moments of distributions of the random variables $\boldsymbol{\eta}_f$ (f = 1, 2,...,7), i.e. $\mathrm{E}(\boldsymbol{\eta}_f) = \mathbf{0}$ and

$\mathrm{Cov}(\boldsymbol{\eta}_f) = \mathbf{V}_f$, $\mathrm{Cov}(\boldsymbol{\eta}_f, \boldsymbol{\eta}_{f'}) = \mathbf{0}$ for all $f \neq f'$. Thus the considered dispersion structure of the linear model has the form

$$\mathrm{Cov}(\mathbf{y}) = \sum_{f=1}^{7} \mathbf{D}'_f \mathbf{V}_f \mathbf{D}_f + \sigma_e^2 \mathbf{I}_n . \tag{3.2}$$

It is easy to show that the dispersion matrix (3.2) can be written as $\mathrm{Cov}(\mathbf{y}) = \sum_{f=0}^{7} \gamma_f \mathbf{P}_f$, where $\{\mathbf{P}_f\}$, $f = 0, 1, \ldots, 7$, are a set of pairwise orthogonal matrices summing up to the identity matrix.

The range space $\Re\{\mathbf{P}_f\}$ of $\mathbf{P}_f$ is termed the f-th stratum with $\mathbf{P}_f$ being orthogonal projection into this stratum. It follows that the considered design has an *orthogonal block structure* (cf. Nelder, 1965). So, the model can be analysed using the methods developed for multistratum experiments. In this case, we have 7 main strata, mentioned above in the Introduction, in which stratum analyses may be performed, at least in some cases. The statistical analysis of submodels related to the different strata are based on algebraic properties of stratum information matrices for treatment combinations, which are defined as

$$\mathbf{A}_f = \boldsymbol{\Delta}\mathbf{P}_f\boldsymbol{\Delta}', \; f = 1, 2, \ldots, 7. \tag{3.3}$$

The presented NSPSB designs will be characterized according to their efficiency of an estimation of treatment combination comparisons (called also orthogonal contrasts) in strata with respect to *general balance* property (cf. Houtman and Speed, 1983). *Stratum efficiency factors* (noted by ε_{fi}) for a set of orthogonal contrasts (noted by $\mathbf{c}'_i\boldsymbol{\tau}$) are eigenvalues of the information matrices $\mathbf{A}_f$, $f = 1, 2, \ldots, 7$ with respect to $\mathbf{r}^{\delta}$, where $\mathbf{r}$ is the vector of replications of the treatment combinations and $\mathbf{r}^{\delta} = \mathrm{diag}(r_1, r_2, \ldots, r_v)$. The contrasts are connected with comparisons among main effects of the considered factors and interaction effects between them. Estimability of the contrasts $\mathbf{c}'_i\boldsymbol{\tau}$ can be checked by the condition

$$\mathbf{A}_f \mathbf{p}_h = \varepsilon_{fi}\mathbf{r}^{\delta}\mathbf{p}_i , \tag{3.4}$$

for f = 1, 2, ..., 7; i = 1, 2, ..., v-1, where $\mathbf{p}_i$ are orthogonal eigenvectors of the matrices $\mathbf{A}_f$, corresponding to ε_{fi} $(0 \le \varepsilon_{fi} \le 1)$ and $\mathbf{c}_i = \mathbf{r}^{\delta}\mathbf{p}_i$.

4. Construction methods of nested split-plot × split-block designs

Statistical properties of (as well complete as incomplete) NSPSB designs follow mainly from crossed and nested treatment structures in an experiment and also from constructing methods, especially in incomplete cases (e.g. Ambroży and Mejza I., 2004b). Treatment structures of the factors impose a number and a structure of units (plots) in the experiment (see Chapter 2). It can be noticed that inside each block in superblocks of NSPSB designs there are five plot sizes (the row, the column I, the column II, the whole plot and the subplot), so there are five levels of a precision with which the effects of the various factors are estimated. The precision is strictly connected with efficiency of the estimation of the contrasts (comparisons) of the treatment combinations. It is well known that the efficiency is the highest in a complete (in particular orthogonal, if it exists) design.

Next, statistical properties of subdesigns for the factors generate statistical properties of the resulting design. So, information split into strata about different contrasts is strictly connected with constructing methods of the NSPSB designs.

In the paper we consider situation when at least one of the subdesigns is a nested incomplete block design, in particular a resolvable incomplete block design.

Let $\mathbf{N}_A$ $(s \times R_A b_A)$, $\mathbf{N}_B$ $(t \times R_B b_B)$ and $\mathbf{N}_C$ $(w \times R_C b_C)$ be incidence matrices of resolvable designs for the row treatments, the column I treatments and the column II treatments with respect to blocks, respectively. They are as follows

$$\mathbf{N}_A = \lfloor \mathbf{N}_{A1} \vdots \mathbf{N}_{A2} \vdots \ldots \vdots \mathbf{N}_{AR_A} \rfloor, \quad \mathbf{N}_B = \lfloor \mathbf{N}_{B1} \vdots \mathbf{N}_{B2} \vdots \ldots \vdots \mathbf{N}_{BR_B} \rfloor,$$

$$\mathbf{N}_C = \lfloor \mathbf{N}_{C1} \vdots \mathbf{N}_{C2} \vdots \ldots \vdots \mathbf{N}_{CR_C} \rfloor, \tag{4.1}$$

where R_A (≥ 1), R_B (≥ 1) and R_C (≥ 1) are numbers of superblocks each of b_A, b_B and b_C blocks with block sizes $\mathbf{k}_1 = k_1\mathbf{1}_{b_A}$, $\mathbf{k}_2 = k_2\mathbf{1}_{b_B}$ and $\mathbf{k}_3 = k_3\mathbf{1}_{b_C}$, in the subdesigns respectively.

The NSPSB design for $v = stw$ treatment combinations in Rb blocks grouped in R superblocks is described by the following incidence matrices: $v \times R$ incidence matrix $\mathbf{N}_1 = \Delta\mathbf{D}'_1$ (with respect to the superblocks), $v \times Rb$ incidence matrix $\mathbf{N}_2 = \Delta\mathbf{D}'_2$ (with respect to the blocks), $v \times Rbk_1$ incidence matrix $\mathbf{N}_3 = \Delta\mathbf{D}'_3$ (with respect to the rows), $v \times Rbk_2$ incidence matrix $\mathbf{N}_4 = \Delta\mathbf{D}'_4$ (with respect to the columns I), $v \times Rbk_2k_3$ incidence matrix $\mathbf{N}_5 = \Delta\mathbf{D}'_5$ (with respect to the columns II), $v \times Rbk_1k_2$ incidence matrix $\mathbf{N}_6 = \Delta\mathbf{D}'_6$ (with respect to the whole plots).

The matrix $\mathbf{N}_1$ can be written as $\mathbf{N}_1 = \Delta\mathbf{D}'_1 = [\mathbf{r}_1 \vdots \mathbf{r}_2 \vdots \cdots \vdots \mathbf{r}_R]$, where $\mathbf{r}_h$ denotes the vector of treatment combinations in the *h-th* superblock. In a designing an experiment the most important role plays the incidence matrix

$$\mathbf{N}_2 = \Delta\mathbf{D}'_2 = [\mathbf{N}_{21} \vdots \mathbf{N}_{22} \vdots \cdots \vdots \mathbf{N}_{2R}], \tag{4.2}$$

where $\mathbf{N}_{2h}$ is an incidence matrix with respect to blocks inside the h-th ($h = 1, 2,\ldots, R$) superblock. It can be noted that

$$\mathbf{N}_1\mathbf{1}_R = \mathbf{N}_2\mathbf{1}_{Rb} = \mathbf{r} = [r_1, r_2, ..., r_v]', \quad \mathbf{N}'_1\mathbf{1}_v = \mathbf{m} = bk_1k_2k_3\mathbf{1}_R,$$

$$\mathbf{N}'_2\mathbf{1}_v = \mathbf{k} = k_1k_2k_3\mathbf{1}_{Rb},$$

where $\mathbf{m}$ is the vector of superblock sizes, and $\mathbf{k}$ is the vector of block sizes in the NSPSB design. Other incidence matrices besides matrix $\mathbf{N}_1$ follow matrix $\mathbf{N}_2$ but their general forms are not unique. However, corresponding to them concurrence matrices, $\mathbf{N}_i\mathbf{N}'_i$, $i = 3, 4, 5, 6$, are unique (see (4.4)).

1) In some situations of a designing experiments *Khatri-Rao product* (called also *semi-Kronecker product*) of submatrices can be used. Khatri and Rao (1968) considered a modification of the ordinary Kronecker product (see also Rao and Mitra, 1971; Gupta and Mukerjee, 1989). Let $\mathbf{N}_A$, $\mathbf{N}_B$ and $\mathbf{N}_C$ be incidence submatrices defined in (4.1). Then according to the incidence matrix (4.1) Khatri-Rao product of these submatrices has a form:

$$\mathbf{N}_2 = \mathbf{N}_A \odot \mathbf{N}_B \odot \mathbf{N}_C =$$

$$[\mathbf{N}_{A1} \otimes \mathbf{N}_{B1} \otimes \mathbf{N}_{C1} \vdots \mathbf{N}_{A2} \otimes \mathbf{N}_{B2} \otimes \mathbf{N}_{C2} \vdots \ldots \vdots \mathbf{N}_{AR} \otimes \mathbf{N}_{BR} \otimes \mathbf{N}_{CR}].$$

Using it in a construction method of the NSPSB design we decrease the number of units R times in the experiment. However, in general it is difficult to obtain and describe statistical properties of a resulting design, only under some assumptions. The method was considered for only some cases of designs for instance, by Mejza et al. (2001), Ambroży and Mejza I. (2004b), Ozawa et al. (2004) and Kuriki et al. (2005).

2) Next method of a designing three-factor experiment is based on the ordinary Kronecker product of submatrices. Then the incidence matrix (4.2) is as follows

$$\mathbf{N}_2 = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{N}_C , \qquad (4.3)$$

where $\mathbf{N}_A$, $\mathbf{N}_B$ and $\mathbf{N}_C$ are given in (4.1). From it, concurrence matrices are of forms:

$$\mathbf{N}_1\mathbf{N}_1' = \sum_{f=1}^{R} \mathbf{r}_f \mathbf{r}_f' , \qquad \mathbf{N}_2\mathbf{N}_2' = \mathbf{N}_A\mathbf{N}_A' \otimes \mathbf{N}_B\mathbf{N}_B' \otimes \mathbf{N}_C\mathbf{N}_C' ,$$

$$\mathbf{N}_3\mathbf{N}_3' = \mathbf{r}_A^\delta \otimes \mathbf{N}_B\mathbf{N}_B' \otimes \mathbf{N}_C\mathbf{N}_C' , \quad \mathbf{N}_4\mathbf{N}_4' = \mathbf{N}_A\mathbf{N}_A' \otimes \mathbf{r}_B^\delta \otimes \mathbf{N}_C\mathbf{N}_C' , \qquad (4.4)$$

$$\mathbf{N}_5\mathbf{N}_5' = \mathbf{N}_A\mathbf{N}_A' \otimes \mathbf{r}_B^\delta \otimes \mathbf{r}_C^\delta , \qquad \mathbf{N}_6\mathbf{N}_6' = \mathbf{r}_A^\delta \otimes \mathbf{r}_B^\delta \otimes \mathbf{N}_C\mathbf{N}_C' ,$$

where $\mathbf{r}_A^\delta = \mathrm{diag}(r_1^A, r_2^A, ..., r_s^A)$, $\mathbf{r}_B^\delta = \mathrm{diag}(r_1^B, r_2^B, ..., r_t^B)$,

$\mathbf{r}_C^\delta = \mathrm{diag}(r_1^C, r_2^C, ..., r_w^C)$.

Then an NSPSB design has R $(= R_A R_B R_C)$ superblocks, each composed of b $(= b_A b_B b_C)$ blocks of equal sizes k $(= k_1 k_2 k_3)$. It is assumed that numbers of replications of the treatment combinations may be different, $\mathbf{r} = \mathbf{r}_A \otimes \mathbf{r}_B \otimes \mathbf{r}_C$.

5. Results

Let $\mathbf{C}_A$, $\mathbf{C}_B$ and $\mathbf{C}_C$ be information matrices for treatments in the subdesigns. They are:

$\mathbf{C}_A = \mathbf{r}_A^\delta - k_1^{-1}\mathbf{N}_A\mathbf{N}_A'$ with nonzero eigenvalues $\mu_1, \mu_2, ..., \mu_{s-1}$ with respect to $\mathbf{r}_A^\delta$,

$\mathbf{C}_B = \mathbf{r}_B^\delta - k_2^{-1}\mathbf{N}_B\mathbf{N}_B'$ with nonzero eigenvalues $\xi_1, \xi_2, ..., \xi_{t-1}$ with respect to $\mathbf{r}_B^\delta$,

$\mathbf{C}_C = \mathbf{r}_C^\delta - k_3^{-1}\mathbf{N}_C\mathbf{N}_C'$ with nonzero eigenvalues $\psi_1, \psi_2, ..., \psi_{w-1}$ with respect to $\mathbf{r}_C^\delta$.

Following algebraic properties of the stratum information matrices for the NSPSB design, $\mathbf{A}_f$, f = 1,2,…, 7 and the information matrices for subdesigns, $\mathbf{C}_A$, $\mathbf{C}_B$ and $\mathbf{C}_C$ one can notice that

– in *the inter-superblock stratum* (f = 1) some contrasts are estimable in the cases only in which an incidence matrix for superblocks, $\mathbf{N}_1$, is not orthogonal, i.e. $\mathbf{N}_1 \neq \frac{\mathbf{rm}'}{n}$. Note that it can be some contrasts among the row treatment effects, among the column I treatment effects and among the interaction effects of the row and the column treatment combinations only or some of them.
  Efficiency factors corresponding to them are equal, respectively, $\varepsilon_{1i} = 1 - \mu_h$, $\varepsilon_{1i} = 1 - \xi_m$ and $\varepsilon_{1i} = (1 - \mu_h)(1 - \xi_m)$. When the NSPSB design is connected and $\mathbf{N}_1 = \frac{\mathbf{rm}'}{n}$ then no information about these contrasts is within this stratum.

– in *the inter-block stratum* (f = 2) some contrasts are estimable in the cases only in which an incidence matrix for the blocks, $\mathbf{N}_2$, is not orthogonal, i.e. $\mathbf{N}_2 \neq \frac{\mathbf{rk}'}{n}$. Then all ($v$ –1) contrasts can be estimable in this stratum or some of them. It depends on subdesigns with incidence matrices $\mathbf{N}_A$, $\mathbf{N}_B$ and $\mathbf{N}_C$, which generate the NSPSB design. Generally, efficiency factors corresponding to these contrasts are equal to
  $\varepsilon_{2i} = 1 - \psi_g$ (for the column II treatments),
  $\varepsilon_{2i} = (1 - \mu_h)(1 - \psi_g)$ (for interaction contrasts of type $A \times C$),
  $\varepsilon_{2i} = (1 - \xi_m)(1 - \psi_g)$ (for interaction contrasts of type $B \times C$)
  $\varepsilon_{2i} = (1 - \mu_h)(1 - \xi_m)(1-\psi_g)$ (for $A \times B \times C$ contrasts).

If $\mathbf{N}_2 = \frac{\mathbf{rk}'}{n}$ then no information about these contrasts is within this stratum.

– If an incidence matrix for the row treatments, $\mathbf{N}_A$ is for a connected and orthogonal subdesign, i.e. $\mathbf{N}_A = \frac{\mathbf{r}_A\mathbf{k}_1'}{n_A}$, then all ($s$ – 1) contrasts are estimated in *the row-stratum* (f = 3) only with full efficiency equal to one. In other cases, i.e. if $\mathbf{N}_A \neq \frac{\mathbf{r}_A\mathbf{k}_1'}{n_A}$ then stratum efficiency factor is $\varepsilon_{3i} = \mu_h$. Additionally in this stratum we may expect estimability of interaction contrasts of types $A \times B$, $A \times C$ and $A \times B \times C$. Efficiency factors corresponding to them can be express by $\varepsilon_{3i} = \mu_h(1 - \xi_m)$, $\varepsilon_{3i} = \mu_h(1 - \psi_g)$ and $\varepsilon_{3i} = \mu_h(1 - \xi_m)(1-\psi_g)$, respectively.
– If an incidence matrix for the column I treatments, $\mathbf{N}_B$ is for a connected and orthogonal subdesign, i.e. $\mathbf{N}_B = \frac{\mathbf{r}_B\mathbf{k}_2'}{n_B}$, then all ($t$ – 1) contrasts are estimated in *the column I-stratum* (f = 4) only with full efficiency equal to one. In other cases, i.e. if $\mathbf{N}_B \neq \frac{\mathbf{r}_B\mathbf{k}_2'}{n_B}$ then stratum efficiency factor is $\varepsilon_{4h} = \xi_m$. Additionally, in this stratum we may expect estimability of interaction contrasts of types $A \times B$, $B \times C$ and $A \times B \times C$. Efficiency factors corresponding to them can be express by $\varepsilon_{4i} = (1 - \mu_h)\xi_m$, $\varepsilon_{4i} = \xi_m(1 - \psi_g)$ and $\varepsilon_{4i} = (1-\mu_h)\xi_m(1-\psi_g)$, respectively.
– If an incidence matrix for column II treatments, $\mathbf{N}_C$ is for a connected and orthogonal subdesign, i.e. $\mathbf{N}_C = \frac{\mathbf{r}_C\mathbf{k}_3'}{n_C}$, then all w – 1 contrasts are estimated in *the column II-stratum* (f = 5) only with full efficiency equal to one. In other case, i.e. if $\mathbf{N}_C \neq \frac{\mathbf{r}_C\mathbf{k}_3'}{n_C}$ then stratum efficiency factor is $\varepsilon_{5i} = \psi_g$. Additionally, in this stratum we may expect estimability of interaction contrasts of types $A \times C$, $B \times C$ and $A \times B \times C$. Efficiency factors corresponding to them can be express by $\varepsilon_{5i} = (1 - \mu_h)\psi_g$, $\varepsilon_{5h} = \psi_g$ and $\varepsilon_{5i} = (1 - \mu_h)\psi_g$, respectively.
– If incidence matrices for the row treatments, $\mathbf{N}_A$, and for the column I treatments, $\mathbf{N}_B$ simultaneously are for orthogonal subdesigns, i.e. their forms can be expressed as below, then all (s – 1)(t – 1) interaction $A \times B$ type contrasts are estimated in *the whole plot-stratum* (f = 6) only with full efficiency equal to one. In other cases the stratum efficiency factor can be cal-

culated from $\varepsilon_{6h} = \mu_h\xi_m$. Additionally, in this stratum we may expect estimability of interaction contrasts of type $A \times B \times C$. Efficiency factors for them are $\varepsilon_{6h} = \mu_h\xi_m(1\text{-}\psi_g)$.

– An analysis within *the sub-plot-stratum* ($f = 7$) is connected with estimation of $(s – 1)(w – 1)$ interaction contrasts of type $A \times C$ and $(s – 1)(t – 1)(w – 1)$ contrasts of type $A \times B \times C$ only. It results from crossed and nested treatment structures of NSPSB designs. However, their efficiencies with respect to these contrasts are strictly connected with subdesigns for the row, column I and column II treatments. Generally, stratum efficiency factors for these contrasts are equal respectively to $\varepsilon_{7h} = \mu_h\psi_g$ and $\varepsilon_{7h} = \mu_h\xi_m\psi_g$. In particular, when incidence matrices $\mathbf{N}_A$ and $\mathbf{N}_C$ are for orthogonal subdesigns, regardless of the incidence matrix $\mathbf{N}_B$, the stratum efficiency factors for $A \times C$, and $A \times B \times C$ type contrasts are equal to one.

6. Examples

To illustrate the theory presented in the paper, consider some $2 \times 5 \times 2$ factorial experiment. Assume that the row treatments correspond to two levels of nitrogen fertilization ($s = 2$), the column I treatments correspond to five varieties of wheat ($t = 5$) and the column II treatments correspond to an application (or not) of a chemical preparat – growth regulator ($w = 2$). According to a material structure of the experiment a resolvable block design for the column I treatments can be used. It will generate an NSPSB design. We will present two examples.

***Example* 6.1.** Consider at first some equireplicate generating design. Following the paper by Caliński and Kageyama (2000b) dual design to a singular group divisible design S51 (see Clatworthy, 1973) with incidence matrix $\tilde{\mathbf{N}}$ can be adopted. It is known that dual with incidence matrix $\tilde{\mathbf{N}}'$ is a resolvable block design.

Consider the NSPSB design described by the incidence matrix (4.3) with

$$\mathbf{N}_A = \mathbf{N}_C = \mathbf{1}_2 \quad \text{and} \quad \mathbf{N}_B\,(=\tilde{\mathbf{N}}') = [\mathbf{N}_{B1} : \mathbf{N}_{B2}],$$

where

Superblock 1 *Superblock 2*

$$\mathbf{N}_{B1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}, \qquad \mathbf{N}_{B2} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Note that the subdesign for the factor B is a resolvable BIB design with parameters

$$t = 5,\ R_B = 2,\ b_B = 5,\ k_2 = 4,\ r_B = 8,\ \lambda = 6,$$

$$\xi_1 = \xi_2 = \xi_3 = \xi_4 = 15/16 = 0.9375.$$

Note also that it generates the resulting design for the considered three-factor experiment. Hence the NSPSB design has two superblocks (R = 2), each composed of five blocks (b = 5) of equal sizes, k (= $k_1k_2k_3$) = 16. Number of the treatment combinations $v = stw = 20$ (> k), each equireplicated, so the vector of replications is $\mathbf{r} = 8\mathbf{1}'_{20}$. An incidence matrix for the superblocks has the form $\mathbf{N}_1 = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{bmatrix}'$, so it is of an orthogonal design.

If $\{A_1, A_2 \mid B_k, B_l, B_m, B_n \mid C_1, C_2\}$, $k < l < m < n$; k, l, m, n = 1,2,3,4,5 denotes the block in which the row treatments A_1, A_2 are allocated to the rows, the column treatments B_k, B_l, B_m, B_n are allocated to the columns I and the subplot treatments C_1, C_2 are allocated to the columns II, then the layout (before randomization) of the $2 \times 5 \times 2$ factorial experiment arranged in the nested SPBB design will be as follows:

Superblock 1	*Superblock 2*
$\{A_1, A_2 \mid B_1, B_3, B_4, B_5 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_2, B_3, B_4, B_5 \mid C_1, C_2\}$,
$\{A_1, A_2 \mid B_1, B_2, B_4, B_5 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_1, B_3, B_4, B_5 \mid C_1, C_2\}$,
$\{A_1, A_2 \mid B_1, B_2, B_3, B_5 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_1, B_2, B_4, B_5 \mid C_1, C_2\}$,
$\{A_1, A_2 \mid B_1, B_2, B_3, B_4 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_1, B_2, B_3, B_5 \mid C_1, C_2\}$,
$\{A_1, A_2 \mid B_2, B_3, B_4, B_5 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_1, B_2, B_3, B_4 \mid C_1, C_2\}$.

The information matrices are of the forms:

$\mathbf{A}_1 = \mathbf{0}$, $\mathbf{A}_2 = \frac{1}{8}\mathbf{J}_2 \otimes (\mathbf{I}_5 - \frac{1}{5}\mathbf{J}_5) \otimes \mathbf{J}_2$,

$\mathbf{A}_3 = \frac{1}{8}(2\mathbf{I}_2 - \mathbf{J}_2) \otimes (\mathbf{I}_5 + 3\mathbf{J}_5) \otimes \mathbf{J}_2$, $\mathbf{A}_4 = \frac{1}{8}\mathbf{J}_2 \otimes (15\mathbf{I}_5 - 3\mathbf{J}_5) \otimes \mathbf{J}_2$,

$\mathbf{A}_5 = \mathbf{J}_2 \otimes 2\mathbf{I}_5 \otimes (2\mathbf{I}_2 - \mathbf{J}_2)$, $\mathbf{A}_6 = \frac{1}{8}(2\mathbf{I}_2 - \mathbf{J}_2) \otimes (15\mathbf{I}_5 - 3\mathbf{J}_5) \otimes \mathbf{J}_2$,

$\mathbf{A}_7 = (2\mathbf{I}_2 - \mathbf{J}_2) \otimes 2\mathbf{I}_5 \otimes (2\mathbf{I}_2 - \mathbf{J}_2)$, where $\mathbf{J}_x = \mathbf{1}_x \mathbf{1}'_x$.

Table 1. Stratum efficiency factors of the NSPSB design – example 6.1

Types of contrasts	Df	Strata						
		(1)	(2)	(3)	(4)	(5)	(6)	(7)
A	1			1				
B	4		0.0625		0.9375			
C	1					1		
$A \times B$	4			0.0625			0.9375	
$A \times C$	1							1
$B \times C$	4					1		
$A \times B \times C$	4							1

Summing up, information connected with estimation of four contrasts among effects of the varieties of wheat (the factor B) is split into two strata. The same situation is for the interaction contrasts of type $A \times B$ (among the nitrogen fertilization and the varieties). However loss of information is not large (≈ 6 %) so the stratum analyses can be performed, for the main effects of the factor B in the column I-stratum and for the interaction $A \times B$ in the whole-plot-stratum (as in a complete design). Other contrasts of types A, C, $A \times C$, $B \times C$ and $A \times B \times C$ are estimated as in a complete design (with full efficiency).

***Example* 6.2**. As generating design for the column I treatments a supplemented block design now will be used. Its incidence matrix has the form $\mathbf{N}_B = [\mathbf{N}_{B1} \vdots \mathbf{N}_{B2} \vdots \mathbf{N}_{B3}]$, where

$$\mathbf{N}_{B1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{N}_{B2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{N}_{B3} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Note that the subdesign for factor B can be treated as a nested block design with parameters

$t = 5$, $R_B = 3$, $b_B = 2$, $k_2 = 3$, $\mathbf{r}_B = [6, 6, 2, 2, 2]'$, $\xi_1 = \xi_2 = 2/3 = 0{,}67$, $\xi_3 = \xi_4 = 1$. Hence, the resulting NSPSB design has two superblocks (R = 3), each composed of five blocks (b = 2) of equal sizes, k (= $k_1 k_2 k_3$) = 12. Number of the treatment combinations $v = stw$ = 20 (> k). They are nonequireplicated, so the vector of replications is $\mathbf{r} = \mathbf{1}_2' \otimes [6, 6, 2, 2, 2]' \otimes \mathbf{1}_2'$. An incidence matrix for the superblocks has now a form $\mathbf{N}_1 = \begin{bmatrix} 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 & 1 \end{bmatrix}$, so it is not of an orthogonal design.

We may expect that some contrasts will be estimated in the inter-superblock stratum. The incidence matrix with respect to the blocks is

$$\mathbf{N}_2 = \mathbf{1}_2 \otimes \mathbf{N}_B \otimes \mathbf{1}_2 .$$

Then the layout (before randomization) of the $2 \times 5 \times 2$ factorial experiment arranged in the resolvable SPBB design will be as follows:

Superblock 1	Superblock 2	Superblock 3
$\{A_1, A_2 \mid B_1, B_2, B_3 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_1, B_2, B_4 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_1, B_2, B_5 \mid C_1, C_2\}$
$\{A_1, A_2 \mid B_1, B_2, B_3 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_1, B_2, B_4 \mid C_1, C_2\}$,	$\{A_1, A_2 \mid B_1, B_2, B_5 \mid C_1, C_2\}$

Information matrices are of the form:

$$\boldsymbol{A}_1 = \mathbf{J}_2 \otimes \begin{bmatrix} \mathbf{0}_{2\times2} & \mathbf{0}_{2\times3} \\ \mathbf{0}_{3\times2} & (3/18)\mathbf{I}_3 - (1/18)\mathbf{J}_3 \end{bmatrix} \otimes \mathbf{J}_2, \qquad \boldsymbol{A}_2 = \mathbf{0},$$

$$\boldsymbol{A}_3 = (2\mathbf{I}_2 - \mathbf{J}_2) \otimes \begin{bmatrix} (1/2)\mathbf{I}_2 & (1/6)\mathbf{J}_{2\times3} \\ (1/6)\mathbf{J}_{3\times2} & (1/6)\mathbf{I}_3 \end{bmatrix} \otimes \mathbf{J}_2,$$

$$\boldsymbol{A}_4 = \mathbf{J}_2 \otimes \begin{bmatrix} (3/2)\mathbf{I}_2 - (1/2)\mathbf{J}_2 & (-1/6)\mathbf{J}_{2\times3} \\ (-1/6)\mathbf{J}_{3\times2} & (1/3)\mathbf{I}_3 \end{bmatrix} \otimes \mathbf{J}_2,$$

$$\boldsymbol{A}_5 = \mathbf{J}_2 \otimes \begin{bmatrix} (3/2)\mathbf{I}_2 & \mathbf{0}_{2\times3} \\ \mathbf{0}_{3\times2} & (1/2)\mathbf{I}_3 \end{bmatrix} \otimes (2\mathbf{I}_2 - \mathbf{J}_2),$$

$$\boldsymbol{A}_6 = (2\mathbf{I}_2 - \mathbf{J}_2) \otimes \begin{bmatrix} (3/2)\mathbf{I}_2 - (1/2)\mathbf{J}_2 & (-1/6)\mathbf{J}_{2\times3} \\ (-1/6)\mathbf{J}_{3\times2} & (1/3)\mathbf{I}_3 \end{bmatrix} \otimes \mathbf{J}_2,$$

$$\boldsymbol{A}_7 = (2\mathbf{I}_2 - \mathbf{J}_2) \otimes \begin{bmatrix} (3/2)\mathbf{I}_2 & \mathbf{0}_{2\times3} \\ \mathbf{0}_{3\times2} & (1/2)\mathbf{I}_3 \end{bmatrix} \otimes (2\mathbf{I}_2 - \mathbf{J}_2).$$

Table 2. Stratum efficiency factors of the NSPSB design – example 2

Types of contrast	Df	Strata						
		(1)	(2)	(3)	(4)	(5)	(6)	(7)
A	1			1				
B	2	0.33			0.67			
	2				1			
C	1					1		
$A\times B$	2			0.33			0.67	
	2						1	
$A\times C$	1							1
$B\times C$	4					1		
$A\times B\times C$	4							1

From the table 2 it can be seen that the contrasts among main effects of the factor B are estimated with a different precision. Similarly to the example 6.1, information connected with two of them is split into two strata (here in (1)- the superblock stratum and (4) – with the column I stratum). The remaining two of them are estimated with full efficiency (the stratum efficiency factor is equal to 1). The stratum analysis connected with main effects of the factor B should be performed in the stratum (4) only. The same situation is with the contrasts of the interaction effects of type $A\times B$. Other contrasts of types A, C, $A\times C$, $B\times C$ and $A\times B\times C$ are estimated with full efficiency as in a complete design.

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KONSTRUKCJA ZAGNIEŻDŻONYCH NIEKOMPLETNYCH UKŁADÓW SPLIT-PLOT × SPLIT-BLOCK

Streszczenie

Zagnieżdżona struktura blokowa układu pozwala kontrolować kilka źródeł zmienności w eksperymencie. Fakt ten wykorzystany został w planowaniu i analizie doświadczeń z trzema czynnikami z pewną zagnieżdżoną i krzyżową strukturą jednostek. Rozważane tu układy doświadczalne mogą być niekompletne ze względu na poziomy jednego czynnika, dwóch lub wszystkich czynników, które są wtedy aranżowane w zagnieżdżonych, w szczególności rozkładalnych, niekompletnych podukładach blokowych. Wygenerowane końcowe układy nazywane są zagnieżdżonymi układami split-plot × split-block (z jęz. ang NSPSB). Przeprowadzane w nich procesy randomizacyjne jednostek różnego rzędu doprowadzają do powstania ośmiu warstw (podprzestrzeni ortogonalnych), przy czym jedna z nich jest zawsze związana z estymacją średniej eksperymentu, a w pozostałych mogą być przeprowadzone analizy warstwowe. W pracy przedstawiono, które kontrasty i z jaką efektywnością są estymowane w różnych warstwach. Rozważania zostały zilustrowane przykładami konstrukcji układów NSPSB z jednakowymi i różnymi liczbami replikacji kombinacji obiektowych.

Słowa kluczowe: ogólne zrównoważenie, doświadczenia wielowarstwowe, układy zagnieżdżone, rozkładalne układy o blokach niekompletnych, układy split- plot × split- block, warstwowe współczynniki efektywności

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