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# ORTHOGONALITY IN ROW-COLUMN DESIGNS

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**Summary**

In the paper, the row-column designs with some orthogonality properties are considered. New relations and some examples are presented.

**Key words and phrases:** row-column design, strict orthogonality, treatment-row orthogonality, treatment-column orthogonality, connectedness, efficiency factor, commutativity

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## 1. Introduction

Considerations of this paper concern the linear model corresponding to *a row-column design* in which $v$ treatments are allocated on $n$ experimental units arranged in $b_1$ rows and $b_2$ columns. In this model, denoted by the triple

$$\{\boldsymbol{y},\ \mathbf{1}_n\mu + \boldsymbol{\Delta}'\boldsymbol{\gamma} + \boldsymbol{D}_1'\boldsymbol{\beta}_1 + \boldsymbol{D}_2'\boldsymbol{\beta}_2, \sigma^2\boldsymbol{I}_n\},$$

where $\boldsymbol{y}$ is an $(n \times 1)$ observable random vector with an expectation $\mathsf{E}(\boldsymbol{y}) = \mathbf{1}_n\mu + \boldsymbol{\Delta}'\boldsymbol{\gamma} + \boldsymbol{D}_1'\boldsymbol{\beta}_1 + \boldsymbol{D}_2'\boldsymbol{\beta}_2$ and a dispersion matrix $\mathsf{D}(\boldsymbol{y}) = \sigma^2\boldsymbol{I}_n$. The symbols $\mathbf{1}_n$ and $\boldsymbol{I}_n$ denote the $(n \times 1)$ vector of ones and the $(n \times n)$ identity matrix, respectively, and $\sigma^2$ is an unknown positive parameter denoting the variance of random disturbances. Further, $\boldsymbol{\gamma}$ is a $(v \times 1)$ vector of unknown treatment effects (main parameters), the scalar $\mu$ is an unknown parameter denoting the overall mean, $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are $(b_1 \times 1)$ and $(b_2 \times 1)$ vectors of unknown row effects and column effects (nuisance parameters), respectively, and $\boldsymbol{\Delta}'$, $\boldsymbol{D}_1'$ and $\boldsymbol{D}_2'$ are $(n \times v)$, $(n \times b_1)$ and $(n \times b_2)$ known binary matrices, such that $\boldsymbol{\Delta}'\mathbf{1}_v = \boldsymbol{D}_1'\mathbf{1}_{b_1} = \boldsymbol{D}_2'\mathbf{1}_{b_2} = \mathbf{1}_n$.

In addition, let $\boldsymbol{N}_1$ be the *treatment-row incidence matrix*, $\boldsymbol{N}_2$ - the *treatment-column incidence matrix*, and $\boldsymbol{N}_{12}$ - the *row-column incidence matrix.* Furthermore, let $\boldsymbol{r}$ denote the vector of treatment replications, $\boldsymbol{k}_1$ the vector of row sizes, and $\boldsymbol{k}_2$ the vector of column sizes. Then, $\boldsymbol{R}$, $\boldsymbol{K}_1$ and $\boldsymbol{K}_2$ are the diagonal matrices with the components of $\boldsymbol{r}$, $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ on their diagonals, respectively.

For a matrix $\boldsymbol{L}$, let $\boldsymbol{Q}_L = \boldsymbol{I} - \boldsymbol{L}(\boldsymbol{L}'\boldsymbol{L})^-\boldsymbol{L}'$ be the orthogonal projector on the orthocomplement of the column space of $\boldsymbol{L}$, where $(\boldsymbol{L}'\boldsymbol{L})^-$ is a generalized inverse of $\boldsymbol{L}'\boldsymbol{L}$.

A crucial role in the analysis of a row-column design belongs to the $\boldsymbol{C}$-*matrix* defined by

$$\boldsymbol{C} = \boldsymbol{\Delta}\boldsymbol{Q}_{(\boldsymbol{D}_1':\boldsymbol{D}_2')}\boldsymbol{\Delta}' = \boldsymbol{R} - \boldsymbol{N}_1\boldsymbol{K}_1^{-1}\boldsymbol{N}_1' +$$

$$-(\boldsymbol{N}_2 - \boldsymbol{N}_1\boldsymbol{K}_1^{-1}\boldsymbol{N}_{12})(\boldsymbol{K}_2 - \boldsymbol{N}_{12}'\boldsymbol{K}_1^{-1}\boldsymbol{N}_{12})^-(\boldsymbol{N}_2 - \boldsymbol{N}_1\boldsymbol{K}_1^{-1}\boldsymbol{N}_{12})'.$$

The $\boldsymbol{C}$-matrix of the two related block subdesigns is of the following form

$$\boldsymbol{C}_h = \boldsymbol{\Delta}\boldsymbol{Q}_{\boldsymbol{D}_h'}\boldsymbol{\Delta}' = \boldsymbol{R} - \boldsymbol{N}_h\boldsymbol{K}_h^{-1}\boldsymbol{N}_h',$$

which corresponds to block designs with the rows $(h = 1)$ and the columns $(h = 2)$ as blocks. The matrix $\boldsymbol{C}_0$ is defined as

$$\boldsymbol{C}_0 = \boldsymbol{R} - \boldsymbol{r}\boldsymbol{r}'/n.$$

Throughout this paper $h = 1, 2$.

## 2. Definitions

**Definition 1.** A row-column design is said to be *connected* if all treatment contrasts are unbiasedly estimable.

It is well known that a row-column design is connected if and only if $\text{rank}(\boldsymbol{C}) = v - 1$.

A row-column design is said to be *row-connected* if and only if $\text{rank}(\boldsymbol{C}_1) = v - 1$. A row-column design is said to be *column-connected* if and only if $\text{rank}(\boldsymbol{C}_2) = v - 1$.

Let $\boldsymbol{A}$ and $\boldsymbol{A}_h$ denote the matrices $\boldsymbol{A} = \boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{C}\boldsymbol{R}^{-\frac{1}{2}}$ and $\boldsymbol{A}_h = \boldsymbol{R}^{-\frac{1}{2}}\boldsymbol{C}_h\boldsymbol{R}^{-\frac{1}{2}}$.

**Definition 2.** The *canonical efficiency factor* $\varepsilon$ of a row-column design and the *canonical efficiency factor* $\varepsilon_h$ of the $h$th block design are harmonic means of nonzero eigenvalues of the corresponding $\boldsymbol{C}$-matrices with respect to the matrix $\boldsymbol{R}$, i.e.

$$\varepsilon^{-1} = \frac{\text{tr}(\boldsymbol{A}^{+})}{\text{rank}(\boldsymbol{C})} \quad \text{and} \quad \varepsilon_h^{-1} = \frac{\text{tr}(\boldsymbol{A}_h^{+})}{\text{rank}(\boldsymbol{C}_h)},$$

where $\text{tr}(\boldsymbol{L})$ denotes the trace of $\boldsymbol{L}$ and $\boldsymbol{L}^{+}$ is a Moore-Penrose inverse of $\boldsymbol{L}$.

**Definition 3.** A row-column design satisfies the *commutativity* property if $\boldsymbol{A}_1\boldsymbol{A}_2 = \boldsymbol{A}_2\boldsymbol{A}_1$.

**Definition 4.** A row-column design is said to be *strictly orthogonal* if the design matrix for rows adjusted for treatments (i.e. $\boldsymbol{Q}_{\Delta'}\boldsymbol{D}_1'$) and the design matrix for columns adjusted for treatments (i.e. $\boldsymbol{Q}_{\Delta'}\boldsymbol{D}_2'$) are mutually orthogonal, i.e.

$$\boldsymbol{D}_1\boldsymbol{Q}_{\Delta'}\boldsymbol{D}_2' = \boldsymbol{O}.$$

It can easily be proved that the above condition can be rewritten as

$$\boldsymbol{N}_1'\boldsymbol{R}^{-1}\boldsymbol{N}_2 = \boldsymbol{N}_{12}. \tag{2.1}$$

For more results of orthogonality in the row-column design, see, for example, Siatkowski (1993).

**Definition 5.** A row-column design is said to be *treatment-row orthogonal* if the design matrix for treatments adjusted for rows (i.e. $\boldsymbol{Q}_{D_1'}\Delta'$) and the design matrix for rows adjusted for treatments (i.e. $\boldsymbol{Q}_{\Delta'}\boldsymbol{D}_1'$) are mutually orthogonal, i.e.

$$\Delta\boldsymbol{Q}_{D_1'}\boldsymbol{Q}_{\Delta'}\boldsymbol{D}_1' = \boldsymbol{O}.$$

It can easily be proved that the last equality can be rewritten as

$$\boldsymbol{N}_1\boldsymbol{K}_1^{-1}\boldsymbol{N}_1'\boldsymbol{R}^{-1}\boldsymbol{N}_1 = \boldsymbol{N}_1. \tag{2.2}$$

**Definition 6.** A row-column design is said to be *treatment-column orthogonal* if the design matrix for treatments adjusted for columns (i.e. $\boldsymbol{Q}_{D_2'}\Delta'$) and the design matrix for columns adjusted for treatments (i.e. $\boldsymbol{Q}_{\Delta'}\boldsymbol{D}_2'$) are mutually orthogonal, i.e.

$$\Delta\boldsymbol{Q}_{D_2'}\boldsymbol{Q}_{\Delta'}\boldsymbol{D}_2' = \boldsymbol{O}.$$

It can easily be proved that the last condition can be rewritten as

$$\boldsymbol{N}_2\boldsymbol{K}_2^{-1}\boldsymbol{N}_2'\boldsymbol{R}^{-1}\boldsymbol{N}_2 = \boldsymbol{N}_2.$$

**Definition 7.** A row-column design is said to be *ordinary row-column design* if the row-column incidence matrix is of the form $\boldsymbol{N}_{12} = \boldsymbol{1}\boldsymbol{1}'$.

## 3. Results and examples

**Theorem 1.** If a connected row-column design is such that the treatment-row subdesign is orthogonal and the treatment-column subdesign is orthogonal then the row-column design satisfy the *decomposability property*, i.e.

$$\boldsymbol{C} = \boldsymbol{C}_1 + \boldsymbol{C}_2 - \boldsymbol{C}_0. \tag{3.1}$$

**Proof.** Follows by Remark 3 in Goszczurna and Siatkowski (2003) and by Corollary in Baksalary and Siatkowski (1993).

**Theorem 2.** A connected row-column design with the treatment-row subdesign orthogonal property and the treatment-column subdesign orthogonal property fulfills the commutativity property.

**Proof.** Follows by Theorem 4.1 in Baksalary and Shah (1990).

**Example 1.** Let us consider a row-column design with the plan of the form

$$\begin{pmatrix} A & B & C & * \\ B & C & * & A \\ C & * & A & B \\ * & A & B & C \end{pmatrix}.$$

For this row-column design

$$\boldsymbol{N}_1 = \boldsymbol{N}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

thus (2.1) holds, i.e. the treatment-row subdesign is orthogonal and (2.2) holds, i.e. the treatment-column subdesign is orthogonal. Consequently, from Theorem 1, the row-column design satisfy the decomposability property (3.1). However, from Theorem 2, the row-column design fulfills the commutativity property.

**Theorem 3.** If a connected ordinary row-column design is such that the treatment-row subdesign is orthogonal and the treatment-column subdesign is orthogonal then the row-column design is strictly orthogonal.

**Proof.** Follows by Theorem 2 in Goszczurna and Siatkowski (2005).

**Example 2.** Let us consider a row-column design with the plan of the form

$$\begin{pmatrix} A & B & B & C & D \\ D & A & B & B & C \\ C & D & A & B & B \\ B & C & D & A & B \\ B & B & C & D & A \end{pmatrix}.$$

For this row-column design $v = 4$, $\boldsymbol{N}_{12} = \mathbf{1}\mathbf{1}'$,

$$\boldsymbol{C} = \begin{pmatrix} 4 & -2 & -1 & -1 \\ -2 & 6 & -2 & -2 \\ -1 & -2 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix},$$

and $\text{rank}(\boldsymbol{C}) = 3$. So, the ordinary row-column design is connected. Next,

$$\boldsymbol{N}_1 = \boldsymbol{N}_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

therefore, the row-column design is the treatment-row orthogonal and the treatment-column orthogonal. In that case, from Theorem 3, the row-column design is strictly orthogonal.

**Theorem 4.** If a row-column design with $\text{rank}(\boldsymbol{C}) = \text{rank}(\boldsymbol{C}_1) = \text{rank}(\boldsymbol{C}_2)$ has the canonical efficiency factor $\varepsilon = 1$, then the treatment-row subdesign is orthogonal and the treatment-column subdesign is orthogonal.

**Proof.** Follows by Theorem 1 and Remark 2 in Goszczurna and Siatkowski (2003).

**Example 3.** Let us consider a row-column design with the plan of the form (Freemann, 1975)

$$\begin{pmatrix} * & A & * & B & * & C \\ B & * & C & * & A & * \\ * & C & * & A & * & B \\ A & * & B & * & C & * \\ C & B & A & C & B & A \end{pmatrix}.$$

For this row-column design

$$C = C_1 = C_2 = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{pmatrix},$$

and $\mathrm{rank}(C) = \mathrm{rank}(C_1) = \mathrm{rank}(C_2) = 2$. Next,

$$A = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

and $\varepsilon = 1$. Therefore, from Theorem 4, the treatment-row subdesign is orthogonal and the treatment-column subdesign is orthogonal.

**Lemma 1.** If a connected row-column design has the canonical efficiency factor $\varepsilon = 1$, then the treatment-row subdesign is orthogonal and the treatment-column subdesign is orthogonal.

**Lemma 2.** If a connected ordinary row-column design has the canonical efficiency factor $\varepsilon = 1$, then the row-column design is strictly orthogonal.

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## ORTOGONALNOŚĆ UKŁADÓW WIERSZOWO-KOLUMNOWYCH

### Streszczenie

W pracy przedstawiono nowe relacje pomiędzy własnościami silnej ortogonalności a ortogonalnościami podukładów blokowych dla układów wierszowo-kolumnowych. Niektóre wyniki zilustrowano przykładowymi planami układów.

**Słowa kluczowe:** układ wierszowo-kolumnowy, silna ortogonalność, ortogonalność obiektowo-wierszowa, ortogonalność obiektowo-kolumnowa, spójność, współczynnik efektywności, przemienność

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