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ON FISHER-TYPE INEQUALITIES FOR DESIGNS ON REGULAR GRAPHS

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Summary

Among research topics in block designs we encounter some notions and techniques linked to graph theory. We bring a few glimpses of its use with the focus on designs on regular graph. We report here on some results in this area concerning Fisher-type inequality.

Key words and phrases: Fisher-type inequality, partial geometric design, strongly regular graph

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1. Introduction

Spectral graph theory is a profitable tool in the study of block designs. Our primary concern here are designs on pseudo-geometric graphs. At this point, Fisher-type inequality appears to be a subject of considerable interest. The paper is an attempt towards a brief exposition of the major results on this topic.

The remaining sections have the following content. All necessary preliminaries are covered by Section 2. In Section 3 a short study concerning Fisher-type inequalities is performed. Examples are gathered in Section 4.

2. Prerequisites

2.1. Strongly regular graphs

This section is devoted to some preliminaries on graphs. The reader is assumed to have some familiarity with the basic items of graph terminology. For more information, we refer to Chartrand and Lesniak (1996). Let us prepare some notation first. We shall use the symbols J, I to denote, respectively, the all-one matrix and the identity matrix of suitable dimensions. Throughout, all graphs under consideration will be simple, finite and undirected. From now on, G is assumed to be a d-regular graph on v vertices, unless stated otherwise.

We shall use $\overline{G}$ to indicate the *complement* of G. We let $A(G)$ denote the adjacency matrix of G. We call an eigenvalue of adjacency matrix $A(G)$ *restricted* if it corresponds to the eigenvector perpendicular to the all-one vector. We let $\mu_1(G) > \ldots > \mu_s(G)$ denote the distinct restricted eigenvalues of $A(G)$. It is convenient to henceforth denote $\mu_0(G) = d$. Accordingly, the whole spectrum of $A(G)$ is $\mu_0(G) \geq \mu_1(G) > \ldots > \mu_s(G)$. It is worth to note that we allow the possibility $\mu_0(G) = \mu_1(G)$. It is indeed the case whenever G is disconnected. We shall proceed by exhibiting some basic properties of strongly regular graphs (henceforth ***srg***).

A d-regular graph G is said to be *strongly regular* whenever it is not complete and for each pair of distinct vertices there are exactly a or b vertices adjacent to both, according to they are adjacent or not, respectively. Throughout, the parameters of ***srg*** will be denoted by (v, d, a, b). This condition can be restated in terms of an adjacency matrix A as

$$A^2 = (d-b)\,I + (a-b)\,A + bJ. \tag{2.1}$$

In the following proposition we cite some well known results about ***srg***s. Proof follows from direct verification.

Proposition 1 *(i) for a* ***srg*** *G we have $d(d-a-1) = b(v-d-1)$, (ii) G is a* ***srg*** *if and only if the adjacency matrix $A(G)$ has precisely two distinct restricted eigenvalues.*

Note $\mu_1(G)$, $\mu_2(G)$ are readily obtained via quadratic equation (2.1)

$$\mu_1 + \mu_2 = a - b,\ \mu_1\mu_2 = b - d. \tag{2.2}$$

For later purposes, we establish some additional notation. Let denote by K_n a complete graph on n vertices. Given a graph G, by kG we shall mean a union of k copies of G. The graph $G = \overline{kK_n}$, is commonly known as a *complete k-partite graph.*

2.2. Designs on regular graph

We define $\mathcal{D}$ to be a class of proper, binary, equireplicate block designs (with r as the replication number) in which v varietes are to be compared via b blocks of size k.

A design $\delta \in \mathcal{D}$ is called a *design on regular graph* provided that its concurrence matrix $S := NN'$ has precisely two distinct off-diagonal entries λ_1, λ_2, hereafter called *λ-values.*

For any such design there is afforded a d-regular graph G with the adjacency rule as folows: there is an edge between two distinct points whenever the corresponding λ-value equals λ_1. G is reffered to as the *underlying graph.* Thus, an equivalent way to formulate this definition is by the equality

$$S = (r - \lambda_2)\, I + (\lambda_1 - \lambda_2)\, A\,(G) + \lambda_2 J. \tag{2.3}$$

We shall use the notation $\delta \in \mathcal{D}_G\,(\lambda_1, \lambda_2)$ as an abbreviation for: $\delta \in \mathcal{D}$ is a design on regular graph G with the λ-values λ_1, λ_2. Further analysis is facilitated by reference to *the following property* $d\lambda_1 + (v - d - 1)\,\lambda_2 = r\,(k - 1)$.

From (2.3) the latent roots of S are easily seen to be

$$\theta_i = (\lambda_1 - \lambda_2)\,\mu_i\,(G) + r - \lambda_2, \; i = 1, \ldots, s. \tag{2.4}$$

One readily verifies that the all-one vector contributes $\theta_0 = rk$ to the whole spectrum of S. Clearly there is possibility $\theta_0 = \theta_1$. If it is the case, a design is called disconnected. As easily seen from the foregoing, an underlying graph accumulates algebraic features of design.

The following notions are of special interest here.

Definition 2 *A design $\delta \in \mathcal{D}_G\,(\lambda_2 + 1, \lambda_2)$ is called a* regular graph design *(for short: **rgd**). A **rgd** is said to be* partial linear space *(for short: **pls**) if $\lambda_2 = 0$.*

2.3. Partial geometries and partial geometric designs

We proceed to generalize the notion of partial linear spaces by *partial geometries* (***pg***) and *partial geometric designs* (***pgd***). Much of the foundational work on ***pgd***s was due to Bose. Partial geometric designs attract statisticians' attention, due to optimality in a wide-ranging class of designs (see Bagchi and Bagchi (2001), Cheng and Bailey (1991, Th 2.2)).

We introduce the necessary terminology (Bose et al (1976, Th 2.1)).

Definition 3 *A design $\delta \in \mathcal{D}$ is said to be* partial geometric *if for suitable integers $t \geq 1$ and c the following equation is satisfied*

$$NN'N = \theta N + tJ, \quad \theta = r + k + c - t - 1. \tag{2.5}$$

Partial geometric designs with $c = 0$ are distinguished by calling them partial geometries.

Bose et al, among others, (1963), (1976), (1979) conducted an extensive study of partial geometric designs. The following lemma plays an essential role in this development (Bose et al (1976, Th 3.3), van Dam and Spence (2005, Pr 2)).

Lemma 4 *Apart from balanced designs, a connected block design $\delta \in \mathcal{D}$ is a* ***pgd*** *if and only if its concurrence matrix is singular with the only non-zero eigenvalue θ, other than the simple eigenvalue rk.*

Here we state some facts to use them later.

Lemma 4 implies that a ***pgd*** must be connected. Turning now to (2.5), one readily establishes that a partial geometry forms a partial linear space (every pair of distinct varieties is contained in at most one block).

We exhibit now some additional properties of partial geometries, consequent upon Lemma 4.

We shall rely on the following well-known result originally stated by Bose (1963).

Lemma 5 *The underlying graph of* $\mathbf{pg}(r,k,t)$ *is a* ***srg*** *with*

$$\left(k\left(1+(k-1)(r-1)t^{-1}\right), r(k-1), k-2+(r-1)(t-1), rt\right). \tag{2.6}$$

Proof. Combining (2.4) with Lemma 4 we arrive at

$$d = r(k-1), \mu_1 = \theta - r = k - t - 1, \mu_2 = -r. \tag{2.7}$$

The rest follows trivially from (2.2) and Proposition 1. □

A ***srg*** is said to be *pseudo-geometric* if its parameters are of the form (2.6).

As a converse to Lemma 5, we have

Proposition 6 *A* ***pls****, whose underlying graph is pseudo-geometric, is a partial geometry.*

Proof. The lemma follows combining (2.7) and Lemma 4. □

Two supplementary remarks are in order. (i) a pseudo-geometric graph need not correspond to a partial geometry; if there is indeed relevant partial geometry then the point graph is said to be *geometric*, (ii) the complement of pseudo-geometric graph need not be pseudo-geometric.

3. Fisher-type inequalities

We have included here an exposition of some known facts related to Fisher's inequality.

Proposition 7 *Let the underlying graph of* ***pls*** δ *be strongly regular. Then* $b < v$ *implies that* δ *is a* ***pg****.*

Proof. Clearly $b < v$ implies singularity of S_δ. Finally, we use Lemma 4 to get the desired result. □

The following statement is due to Bose and Shrikhande (1979, Th 3.1). It provides a generalized version of Proposition 7.

Theorem 8 *Let* δ *be a design on* ***srg*** *and let* $\lambda_1 > \lambda_2$*,* $b < v$*. Then* δ *is a* ***pgd*** *with*

$$t = d\,(\lambda_1 - \lambda_2)\,\frac{k\,(r - \lambda_1)}{r\,(v - k)} + k\lambda_2, \quad c = d\,(\lambda_1 - \lambda_2)\,\frac{\lambda_1}{r} + (k-1)\,(\lambda_2 - 1)\,.$$

Next, we cite related result due to Ionin and Shrikhande (2002, Th 3.2).

Theorem 9 *Let* $\delta \in \mathcal{D}_G\,(\lambda_1, \lambda_2)$*. If* $(r - \lambda_2)\,/\,(\lambda_2 - \lambda_1)$ *is not a multiple eigenvalue of* $A\,(G)$*, then* $v \le b$*.*

Finally, we will conclude with brief discussion of special topics involving partial geometries. The following criterion treats the existence of ***pls*** having $b < v$.

Corollary 10 *Let* G *be a* ***srg****. If* $\delta \in \mathcal{D}_G\,(1, 0)$ *satisfies* $b < v$ *then* $k = 1 - d/\mu_2 > -\mu_2$*, where* μ_2 *is a multiple eigenvalue of* $A\,(G)$*.*

Some simple examples are worth to be presented here. To this end, we briefly examine the graph $\overline{3K_3}$. From $d = 6$, $\mu_2 = -3$ we get $k = 3$. The necessity result of Corollary 10 establishes that associated ***pls*** (if it exists) must have at least 9 blocks. It is indeed the case. It is commonly known as the *Pappus configuration* and it turns out to be ***pg*** $(3, 3, 2)$.

1	1	1	2	2	2	3	3	3
4	5	6	4	5	6	4	5	6
7	8	9	8	9	7	9	7	8

Take in turn the pentagon (cyclic graph of order 5). Its negative restricted eigenvalue is irrational, what exludes immediate the case.

For the sake of completness, it is worthwhile to recognize an existence of trivial ($k = 2$) partial geometries. Proposition 6 in van Dam and Spence (2005) characterizes the case. It asserts that *the underlying graph must be of the form* $\overline{2K_r}$ *or* K_{r+1}.

4. Designs on geometric graphs

This note will be concluded with a number of examples.

4.1. Complete Multipartite Graphs

One readily checks that a ***pg*** $(r, k, k-1)$ yields a complete multipartite graph $\overline{kK_r}$. A *group-divisible* design can be characterized as being a design on $\overline{kK_r}$ with $\mu_1 = 0$, $\mu_2 = -r$. According to (2.4), the following types of singularity will occur: (i) $\theta_1 = 0$ (*singular* design), (ii) $\theta_2 = 0$ (*semi-regular* design).

Moreover, each partial geometry having $t = k-1$ turns out to be a *transversal design*, i.e. it meets the following property: the point set admits a partition into equisized groups, each having with every block precisely one point in common.

The *octahedron* $\overline{3K_2}$ corresponds to the ***pls*** δ_1. According to Proposition 7, $b < v$ guarantees that this design (being semi-regular) is a ***pg*** $(2, 3, 2)$. For a second example consider δ_2. As easily seen from Theorem 8, δ_2 (being singular) exemplifies ***pgd***.

δ_1 :

1	1	5	5
2	4	2	4
3	6	6	3

δ_2 :

1	2	1
2	3	3
4	4	5
5	6	6

4.2. Triangular Graphs

We pay here attention to graphs defined by the triple $(2, k, 2)$. These graphs are known as the *triangular* graphs. (2.7) yields $\mu_1 = k - 3$, $\mu_2 = -2$. The concern here is with the graph assigned to $k = 4$. It is the complement of the *Petersen* graph. Two examples are provided below. As a first example consider δ_3 being $\boldsymbol{pg}\,(2,4,2)$. For a second example consider the design whose allocation of treatments to blocks is given by δ_4. A brief inspection reveals that it is a ***pgd***.

δ_3 :

1	1	2	2	3
3	4	4	5	5
9	7	6	8	6
10	8	10	9	7

δ_4 :

1	1	1	2	3	4
2	2	5	3	4	5
3	6	6	7	6	7
4	7	8	8	8	9
5	9	10	10	9	10

.

Let us allow the possibility that a design has repeated blocks. Replacing each block of the design δ_4 by two copies, gives rise to another partial geometric design. It can be easily established that the resulting design has the same parameters $t = 12$, $c = 8$ as the arrangement δ_5 (being just a transversal design on $\overline{5K_2}$).

δ_5 :

1	1	1	1	1	1	2	2	2	2	2	2
3	3	3	4	4	4	3	3	3	4	4	4
5	5	6	5	6	6	5	6	6	5	5	6
7	8	7	8	7	8	7	8	8	7	8	7
10	10	9	9	9	10	9	9	10	10	9	10

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4.3. Latin Square Type Graphs

A partial geometry with $t = r - 1$, where $2 \leq r < k + 1$, is known as a *net* of order r and deficiency $k - t$. Nets (also called *square lattice designs*) can be characterized as follows.

Given a square array of size k containing k^2 distinct symbols and $r-2$ mutually orthogonal Latin squares of order k, we define a partial geometry with k^2 points (representing the cells of the array) and rk blocks. The blocks are partitioned into r *parallel classes* (each being a partition of the set of points) of k blocks apiece. The first (the second) parallel class corresponds to the rows (respectively the columns) of the array, the remaining $r-2$ families are formed by considering each Latin square in turn, and taking as the blocks the sets of cells containing the same symbol. In extreme case $r=2$ two points are contained in the same block whenever the corresponding cells are in the same row or the same column of the array.

The corresponding underlying graph is called *pseudo Latin square type graph* of type $L_r(k)$. It has the eigenvalues $\mu_1 = k-r$, $\mu_2 = -r$. The special graph $L_2(k)$ is known as the *Lattice graph*. Obviously, in extreme case $r = k+1$, we obtain the complete graph on k^2 vertices. This case has been excluded.

Two examples δ_6, δ_7 follow, the first for $\boldsymbol{pg}(2,3,1)$, while the latter exemplifies $\boldsymbol{pgd}$ on $L_2(3)$.

$$\delta_6 : \begin{array}{|c|}\hline 1\\2\\3\\\hline\end{array}\;\begin{array}{|c|}\hline 4\\5\\6\\\hline\end{array}\;\begin{array}{|c|}\hline 7\\8\\9\\\hline\end{array}\quad\begin{array}{|c|}\hline 1\\4\\7\\\hline\end{array}\;\begin{array}{|c|}\hline 2\\5\\8\\\hline\end{array}\;\begin{array}{|c|}\hline 3\\6\\9\\\hline\end{array}\qquad \delta_7 : \begin{array}{|c|}\hline 1\\2\\3\\4\\5\\6\\\hline\end{array}\;\begin{array}{|c|}\hline 1\\2\\3\\7\\8\\9\\\hline\end{array}\;\begin{array}{|c|}\hline 1\\2\\4\\5\\7\\8\\\hline\end{array}\;\begin{array}{|c|}\hline 1\\3\\4\\6\\7\\9\\\hline\end{array}\;\begin{array}{|c|}\hline 2\\3\\5\\6\\8\\9\\\hline\end{array}\;\begin{array}{|c|}\hline 4\\5\\6\\7\\8\\9\\\hline\end{array}\,.$$

References

Bagchi, B., Bagchi, S. (2001). Optimality of partial geometric designs. *Ann. Stat.* **29**, 577-594.

Bose, R.C. (1963). Strongly regular graphs, partial geometries and partially balanced designs. *Pacific J. Math.* **3**, 389-419.

Bose, R.C., Bridges, W.G., Shrikhande, M.S. (1976). A characterization of partial geometric designs. *Disc. Math.* **16**, 1-7.

Bose, R.C., Shrikhande, M.S. (1979). On a class of partially balanced incomplete block designs, *J. Stat. Plan. Infer.* **3**, 91-99.

Chartrand, G., Lesniak, L. (1996). *Graphs and Digraphs.* Chapman and Hall (third edition).

Cheng, C-S., Bailey, R.A. (1991). Optimality of some two-associate-class partially balanced incomplete-block designs. *Ann. Stat.* **19**, 1667-1671.

van Dam, E.R., Spence, E. (2005). Combinatorial designs with two singular values II. Partial geometric designs. *Lin. Alg. Appl.* **396**, 303-316.

Ionin, Y.J, Shrikhande, M.S. (2002). Fisher's inequality for designs on regular graphs. *J. Stat. Plan. Infer.* **100**, 185-190.

O nierówności Fishera dla układów na grafach regularnych

Streszczenie

Tematem rozważań są układy blokowe na regularnych grafach. W pracy dokonano krótkiego przeglądu znanych rezultatów dotyczących nierówności Fishera w obrębie grafów silnie regularnych, ściśle związanych z takimi obiektami kombinatorycznymi jak układy typu PBIB, geometrie częściowe, układy geometrii częściowych. W szczególności ukazano implikacje wspomnianej nierówności w zakresie współzależności pomiędzy wyodrębnionymi klasami układów. W części ilustracyjnej scharakteryzowano podstawowe rodziny grafów współliniowości dla geometrii częściowych, koncentrując uwagę na układach odznaczających się własnością osobliwości macierzy spotkań.

Słowa kluczowe: graf silnie regularny, nierówność Fishera, układ geometrii częściowej

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