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OPTIMALITY OF CROP ROTATION DESIGNS

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Summary

We are concerned with crop rotation experiments with a test crop. We focus attention on some properties related to A-, E-, D- and MV-optimality. The paper is organized as follows. Section 1 sets the scene in the contexts of crop rotations and contains crucial facts about information matrix of the design. Section 2 covers various kinds of optimality. In section 3 we provide numerical results, practical example and final remarks.

Key words and phrases: optimality, information matrix, crop rotation experiment, crop test

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1. Introduction

This section is divided into three parts containing necessary background. First we discuss some specific aspects of crop rotation experiments (further details can be found in (Przybysz 1982a, Bronowicka-Mielniczuk 2007)). Remaining subsections describe briefly some facts about the linear model and information matrix for treatment contrasts.

1.1. Introduction to crop rotations experiments

Crop rotation is the sequence of cropping where, at least two dissimilar types of crops follow each other on the same land over a period of years. Crop rotation is very valuable cultural control strategy. It maintains soil fertility, reduces soil erosion, helps to control weeds and some insect pests, eliminates the need of chemicals (both pesticides and fertilizers) and in consequence increases net profits. The arrangements of cropping sequences depend on a thorough knowledge of the crops grown and should be based on agro-ecological aspects of the production system (local climate, topography, soils).

Experiments examined here will be assumed to satisfy:

1. Two or more crop rotations are compared via a test crop (one selected species from among all examined plants). Statistical analysis covers test crop yields, only.

2. Duration of the experiment equals to the least common multiple of rotation lengths.

3. The test crop appears in every year at least on one plot within each of compared rotations.

4. Different sequences of species generate various levels of soil fertility in the plots and we treat them as the objects.

5. During the full rotation objects from all compared crop rotations ought to meet each other.

1.2. Linear model

Throughout the paper we consider the linear model (Przybysz 1982b):

$$\mathbf{y} = \mathbf{X}_M\mu + \mathbf{X}_R\rho + \mathbf{X}_A\alpha + \mathbf{X}_1\mathbf{e} + \mathbf{X}_B\beta + \mathbf{X}_{RB}\gamma + \mathbf{X}_{AB}\varphi + \mathbf{X}_2\varepsilon, \qquad (1.1)$$

where $\mathbf{y}$ is an observation vector, μ denotes the mean value, ρ, α, β are the fixed effects of replication, treatment and year, respectively, γ and φ represent interaction effects. $\mathbf{X}_M$ is column vector of ones, $\mathbf{X}_R$, $\mathbf{X}_A$, $\mathbf{X}_B$, $\mathbf{X}_{AB}$ are binary design matrices for blocks, objects, years and interactions, respectively. To complete the specification, we introduce random error components $\mathbf{e}$, ε due to experimental units and technical errors, respectively. Using classical assump-

tions of independent distributions with zero mean and the constant variances σ_1^2, σ_2^2, respectively, the covariance structure of **y** is $Cov\mathbf{y} = \sigma_1^2\mathbf{X}_1\mathbf{X}_1^T + \sigma_2^2\mathbf{I}$.
Model (1.1) can be written as:

$$\mathbf{y} = \mathbf{X\Theta} + \mathbf{X}_1\mathbf{e} + \mathbf{X}_2\varepsilon, \tag{1.2}$$

by using augmented design matrix $\mathbf{X} = [\mathbf{X}_M \vdots \mathbf{X}_R \vdots \mathbf{X}_A \vdots \mathbf{X}_B \vdots \mathbf{X}_{RB} \vdots \mathbf{X}_{AB}]$ and setting $\mathbf{\Theta} = [\mu \vdots \rho \vdots \alpha \vdots \beta \vdots \gamma \vdots \varphi]$. We take the vector $\mathbf{y}$ in order of blocks, objects and years (the fastest moving index corresponds to years).
Under the model (1.2) the coefficient matrix of the reduced normal equations for estimates of treatment effects is $\mathbf{C}_d = \mathrm{R}^{-1}\left(\mathbf{D}(\mathbf{r}) - \frac{1}{k}\mathbf{NN}'\right)$, where $\mathbf{D}(\mathbf{r})$- represents diagonal matrix with $\mathbf{r} = (r_1, r_2, ..., r_v)^T$ along the diagonal, r_i being a number of times the test crop appears on the plot receiving *i*-th treatment (sequence); *k* equals number of plots with the test crop in a single year; **N** is the usual treatments versus years incidence matrix, in which one indicates that a test crop was on the plot, zero refers to the other species; R denotes number of replications. Matrix $\mathbf{C}_d$ is called an information matrix of the design. It is symmetric, non-negative definite matrix with zero row and column sums.

1.2. Treatment contrasts

A treatment contrast is any linear combination $\mathbf{s}^{\mathrm{T}}\alpha$ of treatment effects, where $\mathbf{s}^{\mathrm{T}}\mathbf{1} = 0$. All treatment contrasts are estimable under the design *d*, if and only if $rank(\mathbf{C}_d) = v - 1$, in which case *d* is called connected. From now on, we assume that *d* is connected.

Any generalized inverse of $\mathbf{C}_d$ is a covariance matrix for estimates of treatment contrasts, that is, $var(\mathbf{s}^{\mathrm{T}}\hat{\alpha}) = \sigma^2\mathbf{s}^{\mathrm{T}}\mathbf{C}_d^{-}\mathbf{s}$. Let $0 < \lambda_{d1} \le \lambda_{d2} \le \ldots \le \lambda_{dv-1}$ be the nonzero eigenvalues of $\mathbf{C}_d$. Aside from σ^2, the λ_{di}^{-1} are variances for basic contrasts (John 1987 p 25).

2. Optimality criteria

Optimality criteria for evaluating and comparing different designs are related to estimation of linear parametric functions $\mathbf{s}^{\mathrm{T}}\alpha$. Strictly speaking, we indicate some statistical possessions of the designs in the context of maximal precision (in some special sense) with which estimates of treatment comparisons are made. In general, information matrix determines different optimality criteria (denoted by the letters A-average, E-extreme, D-determinant). We also present MV- optimality criterion.

2.1. E-, A- and D-optimality

The main question is "which experimental design is the "best" or which designs have important statistical properties from practical point of view?". Various optimality criteria can be defined in terms of eigenvalues of information matrix $\mathbf{C}_d$.

Letting $\Phi_p(d) = \left(\sum_{i=1}^{v-1} \lambda_{di}^{-p} / (v-1) \right)^{1/p}, p > 0$, we define $\Phi_0(d) = \lim_{p \to 0} \Phi_p(d)$ and $\Phi_\infty(d) = \lim_{p \to \infty} \Phi_p(d)$.

We shall say that a design *d* is Φ_p optimal if it minimizes the $\Phi_p(d)$ values among all the possible designs.

The following cases are of special interest here.

Φ_0: Φ_0 optimality, also known as D-optimality, aims at minimizing $\Phi_0(d) = \prod_{i=1}^{v-1} \lambda_{di}^{-1}$. This kind of optimality is suitable in the case of looking for the shortest confidence interval for an estimate for treatment contrast. A broad discussion of the D-optimality can be found in Shah and Sinha (1989 p. 56). There exists a graph-theoretic formulation of D-criterion.

Φ_1: Φ_1 optimality is called A-optimality. It is well known and easy to establish that: A-optimal design minimizes the average variance of the estimates of all pairwise comparisons.

Φ_∞: The limit criterion $\Phi_\infty(d) = \max_i \{\lambda_{di}^{-1}\} = \dfrac{1}{\lambda_{d1}}$, corresponds to E-optimality. Observe that E-optimality aims at maximizing the minimum eigenvalue of $\mathbf{C}_d$. It follows that E-optimal design minimizes the maximum variance among the estimates of basic contrasts (John 1987 p. 25). This is equivalent to the finding designs which minimize the maximum variance among all estimates obtained for normalized contrasts (Constantine 1987 p. 354).

To complete the picture, we next give a well-known property on the relationship between optimality criteria.

If matrix $\mathbf{C}_d$ is completely symmetric matrix (i.e. its diagonal elements are constant and its off-diagonal elements are constant), then following relation is fulfilled (Shah, Sinha 1989 p. 10). For any $q > p$, if a design *d* is Φ_p – optimal, then design d is Φ_q – optimal.

Thus, we get relation between various optimality criteria. We can say, that for any design *d* with completely symmetric matrix $\mathbf{C}_d$: D-optimality implies A- and E-optimality, as well as A-optimality implies E-optimality.

2.2. MV-optimality

The next criterion does not depend on the information matrix eigenvalues.

A design *d* is called MV-optimal if it minimizes the maximum of the V_{ij}, where $V_{ij} = \left(C_{ii}^{-} + C_{jj}^{-} - C_{ij}^{-} - C_{ji}^{-}\right)\sigma^2$ is the variance of pairwise comparison comprising *i*-th and *j*-th treatment, where C_{ij}^{-} is the *ij*-th entry of a generalized inverse of $\mathbf{C}_d$. Likewise E-optimality, this is a minimax criterion; which aims at minimizing the maximum loss (as measured by variance) for estimating the elementary contrasts.

3. Optimal designs for crop rotation experiments

3.1. Numerical results

Let $\Pi = \langle v_1,...v_m, B_1,..., B_m \rangle$ denotes the experimental design for *m* crop rotations where v_s (s=1,…,m) is rotation length for *s*-th cropping sequence, B_s is „a generating block" describing allocation of the test crop in *s*-th rotation under study. Using standard notation for cyclic designs, treatments are labelled by $0,...,v_s - 1$. We restrict our attention to connected designs. A summary of numerical findings are included in Table 1 and 2, which gives the results for *m* = 2, 3 respectively and *v* = 3, 4, 5; *b* denotes the number of years.

Table 1. Numerical results for *m*= 2 crop rotations

m = 2	v_s, $s=1,\dots,m$	*b*	B_s, $s=1,\dots,m$	$\Phi_{0,d}$	$\Phi_{1,d}$	$\Phi_{\infty,d}$	$maxV_{ij}$
	$v_1=v_2=4$	b = 4	(0),(01)	1.266	1.25	2.366	4
			(0),(012)	.0833	.8452	1.3905	2.75
			(01),(01)	.0313	.6429	1	1.5
			(01),(02)	.0278	.619	1	1.5
			(012),(01)	.0038	.4677	.6508	1.2727
			(012),(02)	.0042	.483	.9045	1.375
			(012),(012)	$.0^3651$	.3512	.375	.7083
	$v_1=3$ $v_2=4$	b = 12	(0),(0)	.0212	.5476	.6667	1.3333
			(0),(01)	$.0^3314$	.2718	.375	.75
			(0),(02)	$.0^3419$	.2996	.5	.75
			(0),(012)	$.0^438$	.2035	.3333	.6667
	$v_1=v_2=5$	b = 5	(0),(01)	2.4299	1.4167	3.2203	4.4
			(0),(02)	2.4299	1.4167	3.2203	4.4
			(0),(013)	.0564	.8822	1.5307	2.8727
			(0),(012)	.0564	.8822	1.5307	2.8727
			(01),(01)	.02	.7222	1.4472	1.7
			(01),(02)	.0128	.6333	.8	1.4
			(02),(02)	.02	.7222	1.4472	1.7
			(013),(01)	$.0^2102$	.4819	.69	1.2792
			(013),(02)	$.0^2115$	.4978	.7985	1.3474
			(012),(01)	$.0^2115$	.4978	.7985	1.3474
			(012),(02)	$.0^2102$	.4819	.4819	1.2792
			(013),(013)	$.0^31$	.367	.4701	.7697
	$v_1=3$ $v_2=5$	b = 15	(0),(0)	.0079	.531	.6667	1.3333
			(0),(01)	$.0^4467$	.2478	.3	.6
			(0),(012)	$.0^5315$	.1734	.2667	.5333
			(0),(013)	$.0^5315$	.1734	.2667	.5333
			(01),(0)	$.0^3169$	.3468	.5	1
			(01),(01)	$.0^5307$	.1713	.2477	.4553
			(01),(012)	$.0^6303$	.1177	.1346	.2578
			(01),(013)	$.0^6303$	.1177	.1346	.2578

Table 2. Numerical results for *m=3* crop rotations

$m=3$	v_s, $s=1,\dots,m$	b	B_s, $s=1,\dots,m$	$\Phi_{0,d}$	$\Phi_{1,d}$	$\Phi_{\infty,d}$	$max V_{ij}$
	$v_1=v_2=v_3=4$	$b=4$	(0),(01),(01)	.0509	.8712	1.809	3
			(0),(01),(02)	.0452	.8485	1.809	2.8333
			(0),(012),(01)	$.0^{2}531$	.7126	1.3379	2.5455
			(0),(012),(02)	$.0^{2}574$	.7264	1.5991	2.5893
			(02),(01),(01)	$.0^{2}165$	.5682	.75	1.25
			(02),(01),(02)	$.0^{2}211$	.6091	1.5	1.5
			(01),(012),(01)	$.0^{3}266$	.4892	.728	1.2143
			(01),(012),(02)	$.0^{3}258$	.4857	.728	1.2143
			(02),(012),(02)	$.0^{3}326$	.5194	1.2844	1.375
			(012),(012),(01)	$.0^{4}442$	.412	.591	1.158
			(012),(012),(02)	$.0^{4}453$	.415	.711	1.188
			(012),(012),(012)	$.0^{5}804$	.345	.375	.694
	$v_1=3$ $v_2=v_3=4$	$b=12$	(0),(0),(0)	$.0^{3}473$	.536	1	1.333
			(0),(0),(01)	$.0^{5}616$	.3278	.5468	1
			(0),(0),(02)	$.0^{5}821$	.356	.927	1.111
			(0),(0),(012)	$.0^{6}635$	.273	.427	.848
			(0),(01),(01)	$.0^{6}22$	.2241	.3125	.625
			(0),(01),(02)	$.0^{6}206$	.2213	.3125	.625
			(0),(012),(01)	$.0^{7}303$	.1886	.3	.6
			(0),(012),(02)	$.0^{7}318$	.1906	.3	.6
			(0),(012),(012)	$.0^{8}509$	.161	.292	.583
			(01),(0),(01)	$.0^{6}435$	.2604	.4694	.8889
			(01),(0),(02)	$.0^{6}489$	.269	.603	.917
			(01),(0),(012)	$.0^{7}608$	.222	.407	.81
			(01),(01),(01)	$.0^{7}204$	.174	.25	.417
			(01),(01),(02)	$.0^{7}196$	.173	.25	.417
			(01),(012),(01)	$.0^{8}326$	.1448	.1985	.3922
			(01),(012),(02)	$.0^{8}336$	.1461	.2427	.4048
			(01),(012),(012)	$.0^{9}585$	.12	.133	.267

3.2. Application

In this section, we discuss briefly the issue of application. In practice, we are primarily concerned with experiments set up to compare different crop rotations. Rotation lengths shall be referenced by subscript indices by initial blocks. For example, we use $(01)_{v_1=4}$ to denote four-course rotation with initial block [1 1 0 0] (one indicates that the test crop is on the plot). Examples of four-field cropping systems are given in Table 3 and Table 4.

Table 3. The arrangement of experiment with four-course crop rotations

year no	A (75% wheat)				B (50% wheat)			
	plot no				plot no			
	1	2	3	4	1	2	3	4
1	**wheat**	bean	**wheat**	**wheat**	**wheat**	rape	bean	**wheat**
2	**wheat**	**wheat**	bean	**wheat**	**wheat**	**wheat**	rape	bean
3	**wheat**	**wheat**	**wheat**	bean	bean	**wheat**	**wheat**	rape
4	bean	**wheat**	**wheat**	**wheat**	rape	bean	**wheat**	**wheat**

Table 4. The arrangement of experiment with four-course crop rotations

year no	A (75% wheat)				C (50% wheat)			
	plot no				plot no			
	1	2	3	4	1	2	3	4
1	**wheat**	bean	**wheat**	**wheat**	**wheat**	potato	**wheat**	pea
2	**wheat**	**wheat**	bean	**wheat**	pea	**wheat**	potato	**wheat**
3	**wheat**	**wheat**	**wheat**	bean	**wheat**	pea	**wheat**	potato
4	bean	**wheat**	**wheat**	**wheat**	potato	**wheat**	pea	**wheat**

In the crop rotations considered wheat is treated as the test crop. According to notation, we are interested in crop rotation experiments described by the following generating blocks $(012)_{v_1=4}$, $(01)_{v_2=4}$ for Table 3 and $(012)_{v_1=4}$, $(02)_{v_2=4}$ for Table 4. For such experiments we obtain (Table 1): $\Phi_{0,d} = 0.0038$, $\Phi_{1,d} = 0.4677$, $\Phi_{\infty,d} = 0.6508$, $maxV_{ij} = 1.2727$, and $\Phi_{0,d} = 0.0042$, $\Phi_{1,d} = 0.483$, $\Phi_{\infty,d} = 0.9045$, $maxV_{ij} = 1.375$, for A-B and A-C experiments, respectively. The arrangement of plants in Table 3 has better statistical properties than arrangement in Table 4, with respect to D-, A-, E- and MV-optimality.

3.3. Final remarks

The study on optimality should be one of the elements of a well-designed crop rotation experiment. Let us notice, that increase in number of observations results in better statistical properties of the design. We observe that for designs with the same number of observations, the distribution of test crop differentiates the results. For example, for 4-fields rotations, scheme $(02)_{v=4}$ dominates $(01)_{v=4}$ in conjunction with plans: $(01)_{v_1=4}$, $(0)_{v_1=4}(01)_{v_2=4}$, $(01)_{v_1=4}(012)_{v_2=4}$, $(0)_{v_1=3}(01)_{v_2=4}$, $(01)_{v_1=3}(01)_{v_2=4}$. We receive the reverse relation with designs: $(012)_{v_1=4}$, $(02)_{v_1=4}(01)_{v_2=4}$, $(01)_{v_1=3}(012)_{v_2=4}$.

References

Bronowicka-Mielniczuk U. (2007). Efektywność układu płodozmianowego. *Colloquium Biometricum* 37.

Constantine G. (1987). *Combinatorial theory and statistical design*. Wiley, New York.

John J.A. (1987). *Cyclic designs*. London, Chapman and Hall.

Przybysz T. (1982a). Schematy eksperymentalne doświadczeń płodozmianowych. *Roczniki Nauk Rolniczych* Seria A 105, 7–15.

Przybysz T. (1982b). Modele matematyczne doświadczeń płodozmianowych. *Roczniki Nauk Rolniczych* Seria A 105, 17–28.

Shah K., Sinha B. (1989). *Theory of optimal designs*. Lecture Notes in Statistics 54, New Jork, Springer-Verlag.

OPTYMALNOŚĆ DOŚWIADCZEŃ PŁODOZMIANOWYCH

Streszczenie

Celem pracy jest wskazanie pewnych statystycznych własności układów doświadczeń płodozmianowych z rośliną testową. W szczególności, zainteresowani jesteśmy A-, E-, D- i MV-optymalnością. Układ pracy jest następujący. W sekcji pierwszej przedstawiono niezbędne informacje o doświadczeniach płodozmianach z rośliną testową oraz przypomniano kluczowe informacje o macierzy informacji układu. Sekcja druga obejmuje różnorodne kryteria optymalności. W sekcji trzeciej zestawiono wyniki numeryczne związane z ideą optymalności dla układów eksperymentalnych doświadczeń płodozmianowych z rośliną testową, podano przykład praktyczny oraz uwagi końcowe.

Słowa kluczowe: optymalność, macierz informacji, eksperyment płodozmianowy, roślina testowa

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