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ON REGULAR E-OPTIMALITY OF SPRING BALANCE WEIGHING DESIGNS

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Summary

The paper deals with the estimation problem of individual weights of objects in spring balance weighing design satisfying the E-optimality criterion. It is assumed that we have several kinds of spring balances with different precisions. In this case the lower bound for the maximum eigenvalue of the inverse of the information matrix of estimators is obtained. The conditions for this lower bound to be attained are given. Moreover, the incidence matrices of balanced incomplete block designs are used to construct regular E-optimal spring balance weighing designs.

Key words and phrases: balanced incomplete block design, E-optimality, spring balance weighing design

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1. Introduction

The optimality of designs plays a main role in the theory of the experimental design. In many papers concerning the optimality, the weighing experiments are considered. In a spring balance, there is only one-pan and any number of objects can be placed on the pan. Then the pointer provides a rearing which represents the total weight of the objects on the pan.

Nowadays, the spring balance weighing design is the name for the experimental design connected not only with a spring balance, but with each experiment in which the results we can describe as the linear combination of unknown weights of objects with coefficients of this combination equal to 1 or 0. In fact, the weighing designs are applicable to a great variety of problems of measurements, not only for weights, but of lengths, voltages, resistances and concentrations of chemicals in mixtures.

Let us suppose we want to determine the unknown weights of p objects in N weighing operations. We assume that recorded observations are independent and there are not systematic errors. One possible design is to weigh each object separately. But by choosing a more complicated weighing design in which several objects are being placed on the pan together, we obtain the "better" weighing design than previously. This technique appeared in a paper by Yates (1935) and was improved and advanced by Hotelling (1944) and Mood (1946).

Of course, an experimenter want to choose a weighing design that is optimal with respect to some condition. In the literature, several criteria are often expressed in terms of the information matrix. One of them is E-optimality, minimizing the maximum eigenvalue of the inverse of the information matrix. The statistical interpretation of E-optimality is the following: the E-optimal design minimizes the maximum variance of the component estimates of the parameters. For various types of optimality we refer to Pukelsheim (1993).

2. The linear model

Let us consider an experiment in which we want to determine unknown weights of p objects using N measurement operations. Suppose that the results of this experiment can be written as

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e},$$

where $\mathbf{y}$ is an $N \times 1$ random vector of the recorded observations, $\mathbf{X} = (x_{ij})$ is an $N \times p$ $(N \geq p)$ matrix of known elements, with $x_{ij} = 1$ or 0 according to if in the i-th measurement operation the j-th object is included or excluded, $\mathbf{w}$ is a $p \times 1$ vector of unknown weights of the objects and $\mathbf{e}$ is an $N \times 1$ random vector of errors. We assume that there are no systematic errors, i.e. $\mathrm{E}(\mathbf{e}) = \mathbf{0}_N$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_N$ denotes the $N \times 1$ column vector of zeros, and $\mathbf{G}$ is an $N \times N$ positive definite matrix of known elements.

The matrix $\mathbf{X}$ is called the design matrix of a spring balance weighing design with the covariance matrix $\sigma^2\mathbf{G}$. Let $\mathrm{M}_{N,p}(0,1)$ be the set of all $N \times p$ binary matrices and $\mathbf{X} \in \mathrm{M}_{N,p}(0,1)$.

The normal equations estimating $\mathbf{w}$ are of the form

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y},$$

where $\hat{\mathbf{w}}$ is the column vector of estimated weights.

A spring balance weighing design is singular or nonsingular depending on whether the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is singular or nonsingular, respectively. From the assumption that $\mathbf{G}$ is positive definite it follows that the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular, i.e. $\mathrm{rank}(\mathbf{X}) = p$.

If $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular the generalized least-squares estimator of $\mathbf{w}$ is given by the formula $\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$. The covariance matrix of $\hat{\mathbf{w}}$ is $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$. The matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix of $\hat{\mathbf{w}}$.

E-optimality can be described in terms of the maximum eigenvalue of the matrix $(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$, $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}]$ or equivalently in terms of the minimum eigenvalue of the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$, $\lambda_{\min}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})$.

The concept of E-optimality was considered in Raghavarao (1971) and Banerjee (1975). In the case of $\mathbf{G} = \mathbf{I}_N$, where $\mathbf{I}_N$ is the $N \times N$ identity matrix, Jacroux and Notz (1983) established the upper bound for $\lambda_{\min}(\mathbf{X}'\mathbf{X})$ and gave some examples of the E-optimal spring balance weighing designs.

In the present paper we consider the experiment in which we have t kinds of spring balances with different precisions. Let n_h denote the number of measurement operations, in which h-th balance is used, $h = 1,2,\ldots,t$. In this case, the covariance matrix of errors is $\sigma^2\mathbf{G}$, where

$$\mathbf{G} = \begin{bmatrix} g_1\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} & \cdots & \mathbf{0}_{n_1}\mathbf{0}'_{n_t} \\ \mathbf{0}_{n_2}\mathbf{0}'_{n_1} & g_2\mathbf{I}_{n_2} & \cdots & \mathbf{0}_{n_2}\mathbf{0}'_{n_t} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{n_t}\mathbf{0}'_{n_1} & \mathbf{0}_{n_t}\mathbf{0}'_{n_2} & \cdots & g_t\mathbf{I}_{n_t} \end{bmatrix}, \qquad (2.1)$$

where $\sum_{h=1}^{t} n_h = N$ and $g_h > 0$, $h = 1, \ldots, t$.

Further, let us suppose that $\mathbf{X}$ be partitioned correspondingly to the matrix $\mathbf{G}$, i.e.

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_t \end{bmatrix} \qquad (2.2)$$

There are two types of problems connected with the E-optimality criterion. One is to determine how small the maximum eigenvalue of $(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$ can be – that is, to determine the lower bound for $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}]$, where $\mathbf{X} \in \mathrm{M}_{N,p}(0,1)$ with the covariance matrix $\sigma^2\mathbf{G}$ given by (2.1). The other is to find design matrices $\mathbf{X}$ for which the lower bound for $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}]$ is attained.

3. E-optimal spring balance weighing design

In this section we give some new results concerning the lower bound for $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}]$ in two separate cases. The first contains even p objects and the second odd p objects.

Lemma 3.1. Let Π be the set of all $p \times p$ permutation matrices and let $\mathbf{M}$ be a $p \times p$ matrix. If $\overline{\mathbf{M}} = \frac{1}{p!}\sum_{\mathbf{P}\in\Pi} \mathbf{P}^{'}\mathbf{M}\mathbf{P}$ then $\overline{\mathbf{M}} = \left(\frac{\mathrm{tr}(\mathbf{M})}{p} - \frac{\mathrm{Q}(\mathbf{M})}{p(p-1)}\right)\mathbf{I}_p + \frac{\mathrm{Q}(\mathbf{M})}{p(p-1)}\mathbf{1}_p\mathbf{1}_p^{'}$, where $\mathrm{tr}(\mathbf{M})$ is the trace of $\mathbf{M}$, $\mathrm{Q}(\mathbf{M})$ denotes the sum of the off diagonal elements of $\mathbf{M}$ and $\mathbf{1}_p$ is the $p \times 1$ vector of ones. Moreover, $\mathrm{tr}(\overline{\mathbf{M}}) = \mathrm{tr}(\mathbf{M})$ and $\mathrm{Q}(\overline{\mathbf{M}}) = \mathrm{Q}(\mathbf{M})$.

The proof of Lemma 3.1 is based on the concept of Jacroux and Notz (1983).

Lemma 3.2. Let $\mathbf{G}$ be of the form (2.1) and $\mathbf{X} \in \mathrm{M}_{N,p}(0,1)$ denote the matrix of the full column rank p given by (2.2).

1. If p is even, then $\lambda_{\max}[(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X})^{-1}] \geq \frac{4(p-1)}{p\,\mathrm{tr}(\mathbf{G}^{-1})}$.

2. If p is odd, then $\lambda_{\max}[(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X})^{-1}] \geq \frac{4p}{(p+1)\,\mathrm{tr}(\mathbf{G}^{-1})}$.

Proof. Let $\overline{\mathbf{M}}$ denote the average of $\mathbf{M} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ over all elements of Π as in Lemma 3.1 for the design matrix $\mathbf{X}$ of the form (2.2) with $\mathbf{G}$ given by (2.1). Then

$$\lambda_{\max}(\overline{\mathbf{M}}^{-1}) \leq \frac{1}{p!}\sum_{\mathbf{P}\in\Pi}\lambda_{\max}[(\mathbf{P}^{'}\mathbf{M}\mathbf{P})^{-1}] = \frac{1}{p!}\sum_{\mathbf{P}\in\Pi}\lambda_{\max}(\mathbf{P}^{'}\mathbf{M}^{-1}\mathbf{P}) = \lambda_{\max}(\mathbf{M}^{-1}) \quad (3.1)$$

because for positive definite matrices, the maximum eigenvalue of the sum of matrices is less than or equal to the sum of the maximum eigenvalues of these matrices. Furthermore, the maximum eigenvalue of matrix is invariant under each permutation of rows and (the same on) columns.

Now, let k_{hi} denote the number of ones in the i-th row of $\mathbf{X}_h$, $1 \leq k_{hi} \leq p$.

By Lemma 3.1. we obtain $\lambda_{\max}(\overline{\mathbf{M}}^{-1}) = p(p-1)\left[\sum_{h=1}^{t}\sum_{i=1}^{n_h} g_h^{-1}k_{hi}(p-k_{hi})\right]^{-1}$.

As we want to minimize $\lambda_{\max}(\mathbf{M}^{-1})$, we should find the maximum value for $\sum_{h=1}^{t}\sum_{i=1}^{n_h} g_h^{-1}k_{hi}(p-k_{hi})$.

If p is even, $\sum_{h=1}^{t}\sum_{i=1}^{n_h} g_h^{-1}k_{hi}(p-k_{hi}) \le \frac{p^2}{4}\mathrm{tr}(\mathbf{G}^{-1})$. (3.2)

Similarly, if p is odd, $\sum_{h=1}^{t}\sum_{i=1}^{n_h} g_h^{-1}k_{hi}(p-k_{hi}) \le \frac{p^2-1}{4}\mathrm{tr}(\mathbf{G}^{-1})$. (3.3)

Hence the lemma. ■

Definition 3.1. Any nonsingular spring balance weighing design with the design matrix $\mathbf{X}$ given by (2.2) and the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2.1) is said to be regular E-optimal if $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}]$ attains the lower bound in Lemma 3.2., i.e.

1. $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}] = \frac{4(p-1)}{p\,\mathrm{tr}(\mathbf{G}^{-1})}$, if p is even,

or

2. $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}] = \frac{4p}{(p+1)\,\mathrm{tr}(\mathbf{G}^{-1})}$, if p is odd.

Remark 3.1. Notice that if the design matrix $\mathbf{X}$ with the covariance matrix $\sigma^2\mathbf{G}$ is regular E-optimal then it is also E-optimal design. But the inverse implication is not true. Moreover, the E-optimal design in the set of design matrices exists but the regular E-optimal design may not exist.

Theorem 3.1. Any nonsingular spring balance weighing design with the design matrix $\mathbf{X}$ of the form (2.2) and the covariance matrix of errors $\sigma^2\mathbf{G}$ given by (2.1) is regular E-optimal

1. for even p, if each row of $\mathbf{X}_h$ contains exactly $\frac{p}{2}$ ones and

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{p}{4(p-1)}\operatorname{tr}(\mathbf{G}^{-1})\mathbf{I}_p + \frac{p-2}{4(p-1)}\operatorname{tr}(\mathbf{G}^{-1})\mathbf{1}_p\mathbf{1}_p',$$

or

2. for odd p, if each row of $\mathbf{X}_h$ contains either $\frac{p-1}{2}$ or $\frac{p+1}{2}$ ones and

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{p+1}{4p}\operatorname{tr}(\mathbf{G}^{-1})\mathbf{I}_p + \frac{1}{4p}[(p-3)\operatorname{tr}(\mathbf{G}^{-1}) + 4\sum_{h=1}^{t} g_h^{-1} m_h]\mathbf{1}_p\mathbf{1}_p'$$

for some $0 \le m_h \le n_h$, where m_h is the number of rows of $\mathbf{X}_h$ with $\frac{p+1}{2}$ ones.

Proof. Since the proofs for even and odd p are similar, we give the proof only for the case of even p. Notice that $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}]$ attains the lower bound in Lemma 3.2. if equalities in (3.1) and (3.2) hold.

The equality in (3.2) holds if and only if $k_{hi} = \frac{p}{2}$ for each $i = 1,\dots,n_h$ and $h = 1,\dots,t$.

It follows easily that $\operatorname{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}) = \frac{p}{2}\operatorname{tr}(\mathbf{G}^{-1})$ and $\mathrm{Q}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}) = \frac{p(p-2)}{4}\operatorname{tr}(\mathbf{G}^{-1})$.

The theorem will be proved if $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \overline{\mathbf{M}}$ in (3.1) and we apply Lemma 3.1. to give the form of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$. ■

For the special case of $\mathbf{G} = \mathbf{I}_N$, we receive the theorem given in Jacroux and Notz (1983).

4. BIB designs leading to regular E-optimal spring balance weighing designs

Now, we will consider the construction of the regular E-optimal spring balance weighing design with matrix $\mathbf{X}$ given by (2.2) and the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2.1) using incidence matrices of the balanced incomplete block designs (BIB designs).

A balanced incomplete block design (see e.g. Raghavarao (1971)) is an arrangement of v treatments in b blocks of size k $(k < v)$, such that each treatment occurs in r blocks and every pair of distinct treatments occurs together in exactly λ blocks. The numbers v, b, r, k, λ are called the parameters of the BIB design and they are related by the following identities $vr = bk$ and $\lambda(v-1) = r(k-1)$. By writing the incidence matrix of the BIB design $\mathbf{N} = (n_{ij})$ where $n_{ij} = 1$ or 0 according as the i-th treatment occurs or does not occur in the j-th block, we have $\mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$.

Let $\mathbf{N}_h$ be the incidence matrix of the balanced incomplete block design with parameters $v, b_h, r_h, k_h, \lambda_h$, $h = 1,2,\ldots,t$. Now, in (2.2) we assume that $\mathbf{X}_h = \mathbf{N}_h'$ and then we have

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' \\ \mathbf{N}_2' \\ \vdots \\ \mathbf{N}_t' \end{bmatrix}. \tag{4.1}$$

In this design $b_h = n_h$ and $v = p$.

Moreover, we assume that the parameters v and b_h satisfy the necessary conditions for the existence of BIB designs.

Lemma 4.1. The existence of BIB designs with the parameters v, b_h and

1. $r_h = \frac{b_h}{2}$, $k_h = \frac{v}{2}$, $\lambda_h = \frac{b_h(v-2)}{4(v-1)}$, if v is even,
2. $r_h = \frac{b_h(v-1)}{2v}$, $k_h = \frac{v-1}{2}$, $\lambda_h = \frac{b_h(v-3)}{4v}$, or

   $r_h = \frac{b_h(v+1)}{2v}$, $k_h = \frac{v+1}{2}$, $\lambda_h = \frac{b_h(v+1)}{4v}$, if v is odd, implies the existence of the regular E-optimal spring balance weighing design with design matrix $\mathbf{X}$ given by (4.1).

Proof. For the design matrix $\mathbf{X}$ of the form (4.1) with $\mathbf{G}$ given by (2.1), we obtain $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \sum_{h=1}^{t} g_h^{-1}\mathbf{N}_h\mathbf{N}_h'$ and $\mathbf{N}_h\mathbf{N}_h' = (r_h - \lambda_h)\mathbf{I}_v + \lambda_h \mathbf{1}_v\mathbf{1}_v'$, $h = 1,\ldots,t$.

Since $\mathbf{N}_h$ is the incidence matrix of the BIB design, it is clear that if v is even, we have $k_h = \frac{v}{2}$, which implies $r_h = \frac{b_h}{2}$ and $\lambda_h = \frac{b_h(v-2)}{4(v-1)}$. An easy computation shows that $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{v}{4(v-1)}\operatorname{tr}(\mathbf{G}^{-1})\mathbf{I}_v + \frac{v-2}{4(v-1)}\operatorname{tr}(\mathbf{G}^{-1})\mathbf{1}_v\mathbf{1}_v'$.

Thus $\mathbf{X}$ is regular E-optimal.

The proof for v odd is similar. ■

Under the conditions given in Lemma 4.1. we formulate the following theorem. For more details about the methods of construction of BIB designs we refer a reader to e.g. Raghavarao (1971), Koukouvinos and Seberry (1997).

Theorem 4.1. The existence of BIB designs with the parameters

1. if v is even,
   i. $v = 2t,\ b = 4t-2,\ r = 2t-1,\ k = t,\ \lambda = t-1,\ t = 2,3,\ldots$
   ii. $v = 4t+4,\ b = 8t+6,\ r = 4t+3,\ k = 2t+2,\ \lambda = 2t+1$ and $4t+3$ is a prime or a prime power
2. if v is odd
   i. $v = 4t-1 = b,\ r = 2t-1 = k,\ \lambda = t-1,\ t = 2,3,\ldots$
   ii. $v = 4t-1 = b,\ r = 2t = k,\ \lambda = t,\ t = 1,2,\ldots$
   iii. $v = 4t+1,\ b = 8t+2,\ r = 4t,\ k = 2t,\ \lambda = 2t-1$ and $4t+1$ is a prime or a prime power
   iv. $v = 4t+1,\ b = 8t+2,\ r = 4t+2,\ k = 2t+1,\ \lambda = 2t+1$ and $4t+1$ is a prime or a prime power

implies the existence of the regular E-optimal spring balance weighing design with design matrix $\mathbf{X}$ given by (4.1).

Proof. It is easy to see that the parameters given in 1. i.- ii. and 2. i.- iv. satisfy the conditions given in Lemma 4.1. ■

5. Examples

In this section we give two examples of two considered cases. In both cases, we assume that we have two balances at our disposal, for example one of higher precision and the other as usual. For the estimation of the individual weights of the objects we want to use the regular E-optimal spring balance weighing design.

Example 5.1. Let us consider the problem of estimating $p = 4$ objects using $N = 12$ measurement operations. The covariance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix $\mathbf{G} = \begin{bmatrix} g_1\mathbf{I}_6 & \mathbf{0}_6\mathbf{0}_6^{'} \\ \mathbf{0}_6\mathbf{0}_6^{'} & g_2\mathbf{I}_6 \end{bmatrix}$, $g_1, g_2 > 0$. To construct the design matrix $\mathbf{X}$ we can use the BIB design with the parameters $v = 4$, $b = 6$, $r = 3$, $k = 2$, $\lambda = 1$, with the incidence matrix

$$\mathbf{N}_1 = \mathbf{N}_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

In this case, we have $\mathbf{X} = \begin{bmatrix} \mathbf{N}_1^{'} \\ \mathbf{N}_2^{'} \end{bmatrix}$, and then

$$\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X} = \frac{\operatorname{tr}(\mathbf{G}^{-1})}{3}\mathbf{I}_4 + \frac{\operatorname{tr}(\mathbf{G}^{-1})}{6}\mathbf{1}_4\mathbf{1}_4^{'} \quad \text{with} \quad \lambda_{\max}[(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X})^{-1}] = \frac{3}{\operatorname{tr}(\mathbf{G}^{-1})},$$

what it means that the conditions from Theorem 3.1. are fulfilled.

Example 5.2. Let us consider the problem of estimating $p = 7$ objects using $N = 14$ measurement operations. The covariance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix $\mathbf{G} = \begin{bmatrix} g_1\mathbf{I}_7 & \mathbf{0}_7\mathbf{0}_7^{'} \\ \mathbf{0}_7\mathbf{0}_7^{'} & g_2\mathbf{I}_7 \end{bmatrix}$, $g_1, g_2 > 0$. To construct the design matrix $\mathbf{X}$ we can use the BIB designs with parameters and incidence matrices, respectively:

$$v = 7 = b_1,\ r_1 = 3 = k_1,\ \lambda_1 = 1; \qquad v = 7 = b_2,\ r_2 = 4 = k_2,\ \lambda_2 = 2$$

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \qquad \mathbf{N}_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

We can obtain the regular E-optimal spring balance weighing design with the design matrix of the form:

i. $\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' \\ \mathbf{N}_1' \end{bmatrix}$, and then $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{2\,\mathrm{tr}(\mathbf{G}^{-1})}{7}\mathbf{I}_7 + \frac{\mathrm{tr}(\mathbf{G}^{-1})}{7}\mathbf{1}_7\mathbf{1}_7'$

ii. $\mathbf{X} = \begin{bmatrix} \mathbf{N}_2' \\ \mathbf{N}_2' \end{bmatrix}$, and then $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{2\,\mathrm{tr}(\mathbf{G}^{-1})}{7}(\mathbf{I}_7 + \mathbf{1}_7\mathbf{1}_7')$

iii. $\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix}$, and then $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{2\,\mathrm{tr}(\mathbf{G}^{-1})}{7}\mathbf{I}_7 + \frac{\mathrm{tr}(\mathbf{G}^{-1}) + 7g_2^{-1}}{7}\mathbf{1}_7\mathbf{1}_7'$

iv. $\mathbf{X} = \begin{bmatrix} \mathbf{N}_2' \\ \mathbf{N}_1' \end{bmatrix}$, and then $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{2\,\mathrm{tr}(\mathbf{G}^{-1})}{7}\mathbf{I}_7 + \frac{\mathrm{tr}(\mathbf{G}^{-1}) + 7g_1^{-1}}{7}\mathbf{1}_7\mathbf{1}_7'$.

In cases i.-iv. we receive $\lambda_{\max}[(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}] = \frac{7}{2\,\mathrm{tr}(\mathbf{G}^{-1})}$, what it means that the conditions from Theorem 3.1. are fulfilled.

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O REGULARNEJ E-OPTYMALNOŚCI SPRĘŻYNOWYCH UKŁADÓW WAGOWYCH

Streszczenie

Praca poświęcona jest zagadnieniu E-optymalnej estymacji nieznanych miar obiektów w modelu sprężynowego układu wagowego przy założeniu, że mamy dostępnych kilka urządzeń pomiarowych o różnej precyzji. W rozważanym modelu zostało podane dolne ograniczenie dla największej wartości własnej macierzy kowariancji estymatora wektora parametrów oraz określono warunki, dla których podane ograniczenie jest osiągnięte. Ponadto opisano metodę konstrukcji regularnych E-optymalnych sprężynowych układów wagowych wykorzystującą macierze incydencji układów zrównoważonych o blokach niekompletnych.

Słowa kluczowe: E-optymalność, sprężynowy układ wagowy, układ zrównoważony o blokach niekompletnych

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