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STATISTICAL PROPERTIES OF SOME SUPPLEMENTED SPLIT–SPLIT–PLOT DESIGN

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Summary

Main purpose of the paper is to provide a new method of the construction of non–orthogonal split–split–plot design for three or more factor experiments. An orthogonally supplemented PEB block design with at most $(m + 1)$ – classes of efficiency generates a new layout. Attention is paid to optimal statistical properties with respect to the efficiency of estimation of some group of the contrasts in the resulting design.

Key words and phrases: control treatments, general balance, supplemented split–split–plot design, stratum efficiency factors

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1. Introduction

In agricultural research the traditional (complete) split–split–plot (SSP) design on RCB design is commonly used to a three–factor experiment. It is an extension of a split–plot design to accommodate a third factor (e.g. Gomez and Gomez, 1984, Section 4.3). The SSP arrangement is characterized as follows:

– three plot sizes corresponding to the three factors, it means the first factor (say, *A*) is assigned to the whole plots, the second factor (say, *B*) to the subplots, and the third factor (say, C) to the sub–subplots.

– there are three levels of precision with the factor C receiving the highest precision.

Such experiments are usually large which is not desirable in practice because of their high cost and complexity. Also, frequently limited amount of the experimental material does not allow using a complete SSP design. Hence, it is worth considering some incomplete SSP designs with respect to at least one of three factors (e.g. Mejza 1997a, 1997b).

In the paper we consider a situation when the incomplete SSP design is orthogonally supplemented by a new group of sub–subplot treatments (called control sub–subplot treatments).

The s*upplemented* (*augmented*) *block designs* for one–factor experiments are described in the literature (e.g. Caliński 1971, Caliński and Ceranka 1974, Singh and Dey 1979, Puri *et al*. 1977, Kachlicka and Mejza 1998, Caliński and Kageyama 2003, Sections 6.3. and 10.3.3). Generally, two sets of treatments exist in all the above designs. Usually one set is referred to as the set of basic (test) treatments and the other – the set of supplementary (control) treatments. The major aim of such experiments is the comparison of both sets of treatments and the treatments inside those sets.

In the paper we present a randomization model, statistical properties and their consequences for an analysis of the resulting design.

2. Material structure

There is assumed the experimental material can be divided into b blocks with k_1 whole plots. Then, each whole plot is divided into k_2 subplots with k_3 sub–subplots. The s levels of factor A (whole plot treatments) are randomly allotted to the whole plots within each block, t levels of factor B (subplot treatments) are randomly allotted to the subplots within each whole plot, and the w levels of factor C (sub–subplot treatments) are randomly allotted to the sub–subplots within each subplot. Hence, the third factor C is in a split–plot relation to the whole plot and subplot treatment combinations (i.e. combinations of the levels of factor A and factor B which are also in a split–plot design).

3. Linear model

As a result of certain assumptions and performed four randomization processes in the experiment the mixed linear model of vector $\mathbf{y}$ of n $(= bk_1k_2k_3)$ observations has the form:

$$\mathbf{y} = \Delta'\boldsymbol{\tau} + \sum_{f=1}^{4} \mathbf{D}'_f \boldsymbol{\eta}_f + \mathbf{e}, \tag{3.1}$$

and the following properties:

$$\mathrm{E}(\mathbf{y}) = \boldsymbol{\Delta}'\boldsymbol{\tau} \quad \mathrm{Cov}(\mathbf{y}) = \sum_{f=1}^{4} \mathbf{D}'_f \mathbf{V}_f \mathbf{D}_f + \sigma_e^2 \mathbf{I}_n, \tag{3.2}$$

where $\boldsymbol{\Delta}'$ is a known design matrix for $v = stw$ treatment combinations, $\boldsymbol{\tau}$ ($v \times 1$) is the vector of fixed treatment combination effects, $\mathbf{D}_1^{'}$, $\mathbf{D}_2^{'}$, $\mathbf{D}_3^{'}$, $\mathbf{D}_4^{'}$ are respectively, ($n \times b$), ($n \times bk_1$), ($n \times bk_1k_2$), ($n \times bk_1k_2k_3$) – design matrices for blocks, the whole plots (within the blocks), the subplots (within the whole plots inside the blocks), and the sub–subplots (within the subplots inside the whole plots and blocks). They are expressed by:

$$\mathbf{D}'_1 = \mathbf{I}_b \otimes \mathbf{1}_{k_1} \otimes \mathbf{1}_{k_2} \otimes \mathbf{1}_{k_3}, \qquad \mathbf{D}'_2 = \mathbf{I}_b \otimes \mathbf{I}_{k_1} \otimes \mathbf{1}_{k_2} \otimes \mathbf{1}_{k_3},$$

$$\mathbf{D}'_3 = \mathbf{I}_b \otimes \mathbf{I}_{k_1} \otimes \mathbf{I}_{k_2} \otimes \mathbf{1}_{k_3}, \qquad \mathbf{D}'_4 = \mathbf{I}_b \otimes \mathbf{I}_{k_1} \otimes \mathbf{I}_{k_2} \otimes \mathbf{I}_{k_3} = \mathbf{I}_n,$$

where $\mathbf{I}_x$ is the identity matrix of order x, $\mathbf{1}_x$ is the x–dimensional vector of ones, $\mathbf{J}_x = \mathbf{1}_x\mathbf{1}'_x$, and $\otimes$ denotes Kronecker product of matrices.
The $\boldsymbol{\eta}_f$ (f = 1, 2, 3, 4) are, respectively, random effect vectors of the blocks, the whole plots, the subplots, the sub–subplots with $\mathrm{E}(\boldsymbol{\eta}_f) = \mathbf{0}$, and $\mathrm{Cov}(\boldsymbol{\eta}_f) = \mathbf{V}_f$, $\mathrm{Cov}(\boldsymbol{\eta}_f, \boldsymbol{\eta}_{f'}) = \mathbf{0}$ for all $f \neq f'$. If σ_f^2 (f = 1, 2, 3, 4) define variances of the variables $\boldsymbol{\eta}_f$, then

$$\mathbf{V}_1 = \sigma_1^2 (\mathbf{I}_b - b^{-1}\mathbf{J}_b), \qquad \mathbf{V}_2 = \sigma_2^2 \mathbf{I}_b \otimes (\mathbf{I}_{k_1} - k_1^{-1}\mathbf{J}_{k_1}), \tag{3.3}$$

$$\mathbf{V}_3 = \sigma_3^2 \mathbf{I}_{bk_1} \otimes (\mathbf{I}_{k_2} - k_2^{-1}\mathbf{J}_{k_2}), \quad \mathbf{V}_4 = \sigma_4^2 \mathbf{I}_{bk_1k_2} \otimes (\mathbf{I}_{k_3} - k_3^{-1}\mathbf{J}_{k_3}) .$$

According to the assumed orthogonal block structure of the considered SSP design, the covariance matrix (3.2) can be written as $\mathrm{Cov}(\mathbf{y}) = \sum_{f=0}^{4} \gamma_f \mathbf{P}_f$, where

$\gamma_f \geq 0$ and the $\mathbf{P}_f$ matrices form a complete known binary set of matrices defining strata of the blocking structure of the design (cf. Mejza, 1997a). More, the range space $\Re\{\mathbf{P}_f\}$ of $\mathbf{P}_f$, f = 0, 1, 2, 3, 4 is termed the f–th stratum of the model, and $\{\gamma_f\}$ are unknown stratum variances (cf. Houtman and Speed, 1983).

In the SSP model there are five strata, i.e. the total area stratum (zero stratum), the inter–block stratum (the first stratum), the inter–whole plot stratum (the second stratum), the inter–subplot stratum (the third stratum) and the inter–sub–subplot stratum (the fourth stratum).

The orthogonal block structure of the considered design allows one to apply Nelder's approach to the analysis of variance for the multistratum experiments (Nelder 1965a, 1965b). The stratum analyses are expressed in terms of basic contrasts introduced by Pearce *et al*. (1974). They are generated by $\mathbf{r}^{\delta}$ – orthonormal eigenvectors of stratum information matrices for the treatment combinations, $\mathbf{A}_f = \Delta \mathbf{P}_f \Delta'$, where $\mathbf{r}^{\delta} = \mathrm{diag}[r_1, r_2, ..., r_v]$ and r_h denotes replicate of the h –th treatment combination, f = 0, 1, 2, 3, 4; $h = 1, 2, ..., v$. General forms of the $\mathbf{A}_f$ for the incomplete and complete SSP experiment designs are given in Mejza (1997a). Their forms appropriate for the considered SSP design one can find in section 4. In the paper we assume that SSP design is generally balanced (cf. Houtman and Speed, 1983). General balance (GB) property occurs when all matrices $\mathbf{A}_f$ fulfill the following criterion (e.g. Mejza, 1992):

$$\mathbf{A}_f \mathbf{r}^{-\delta} \mathbf{A}_{f'} = \mathbf{A}_{f'} \mathbf{r}^{-\delta} \mathbf{A}_f , \tag{3.4}$$

for $f, f' = 1, 2, 3, 4$, $f \neq f'$ and $\mathbf{r}^{-\delta} = \mathrm{diag}[1/r_1, 1/r_2, ..., 1/r_v]$.

The GB property allows finding common set of $\mathbf{r}^{\delta}$– orthonormal eigenvectors for all the information matrices $\mathbf{A}_f$. Eigenvalues, say ε_{fh}, corresponding to the eigenvectors of the matrices $\mathbf{A}_f$ with respect to $\mathbf{r}^{\delta}$ are called stratum efficiency factors. They satisfy the following relations:

$$0 \leq \varepsilon_{fh} \leq 1 , \qquad \underset{h<v}{\forall} (\varepsilon_{0v} = 1,\ \varepsilon_{0h} = 0), \qquad \sum_{f=1}^{4} \varepsilon_{fh} = 1,$$

$$\text{for} \quad h < v ; \quad f = 0, 1, ..., 4 ; \quad h = 1, 2, ..., v .$$

4. Construction method

In this chapter some method of constructing the incomplete SSP design ($k_1 = s$, $k_2 = t$, $k_3 < w$) is described, along with some statistical properties, mainly those related to the efficiency factors of the design for estimating the corresponding basic contrasts in the stratum analyses.

This method is based on the Kronecker product of three designs, in which the levels of three factors (*A*, *B*, *C*) are assigned. Consider situation when the whole plot (*A*) treatments and the subplot (*B*) treatments are in appropriate RCB designs whereas the sub–subplot treatments (*C*) occur in a supplemented block design $d^*(v^* = w,\ b^*,\ k^*,\ \mathbf{r}^*)$, where the parameters v^*, b^*, k^* mean numbers of the sub–subplot treatments, blocks, units inside each block in the subdesign d^*, respectively and $\mathbf{r}^*$ denotes a vector of replicates of the sub–subplot treatments.

We assume the sub–subplot (*C*) treatments consist of two groups: $w = w_1 + w_2$, where w_1 test (basic) C treatments are allocated in a subdesign $\tilde{d}_1$ which is a partially efficiency balanced (PEB) design with at most *m* efficiency classes (cf. Puri *et al.* 1977, Kageyama and Puri 1985, Caliński and Kageyama 2000, Definition 4.3.1.) while w_2 additional (control) C treatments – in a subdesign $\tilde{d}_2$ represented by an orthogonal block design (cf. Caliński and Kageyama, 2000, Definitions 2.2.7–2.2.8).

Let $\tilde{\mathbf{N}}_1$ be the $w_1 \times b_1$ incidence matrix of the subdesign $\tilde{d}_1$ with parameters w_1, b_1, $\tilde{k}_1$, $\mathbf{r}_{w_1}$, ε_j, ρ_j ($\sum_{j=1}^{m} \rho_j = w_1 - 1$). Then (cf. Puri and Nigam 1977, Nigam and Puri 1982, Caliński and Kageyama 2003, e.g. Theorems 6.3.1. and 10.3.3.):

$$\mathbf{N}_{d^*} = \begin{bmatrix} \tilde{\mathbf{N}}_1 \\ \mathbf{r}_{w_2}(\mathbf{k}^*)'/n^* \end{bmatrix}, \tag{4.1}$$

is the incidence matrix of the PEB design with at most (m + 1)–classes of efficiency with parameters:

$$v^* = w = w_1 + w_2,\ \ b^* = b_1,\ \ \ \mathbf{k}^* = n^* \tilde{k}_1 \mathbf{1}_{b_1} / \tilde{n}_1,$$

$$\mathbf{r}^* = \mathbf{N}_{d^*}\mathbf{1} = [\mathbf{r}'_{w_1}, \mathbf{r}'_{w_2}]', \ \varepsilon_0^* = 1, \ \rho_0^* = w_2, \tag{4.2}$$

$$\rho_j^* = \rho_j, \quad j = 1, 2, ..., m,$$

where $\tilde{n}_1$ and $n*$ denote numbers of observations in design $\tilde{d}_1$ and d^*, respectively.

Let $\mathbf{N}_1 = \mathbf{1}_{st} \otimes \mathbf{N}_{d^*}$ be the $v \times b$ incidence matrix of the considered SSP design with parameters $v = st(w_1 + w_2)$, $b = b^*$, $k = stk^*$, $\mathbf{r} = \mathbf{1}_{st} \otimes \mathbf{r}^*$, $n = bstk^*$, where $\mathbf{N}_{d^*}$ is given in (4.1). This method of the construction yields proper (cf. Caliński and Kageyama, 2000, Definition 2.2.2) and non–equireplicated experiment SSP design (cf. Caliński and Kageyama, 2000, Definition 2.2.3).

As mentioned in section 3 statistical properties of the design are related mainly to algebraic properties of the stratum information matrices $\mathbf{A}_f$, which forms in this case are following:

$$\mathbf{A}_1 = \mathbf{C}_0 - \mathbf{C}_1, \ \mathbf{A}_2 = \mathbf{C}_1 - \mathbf{C}_2, \ \mathbf{A}_3 = \mathbf{C}_2 - \mathbf{C}_3, \ \mathbf{A}_4 = \mathbf{C}_3, \tag{4.3}$$

where, assuming $(\mathbf{r}^*)^\delta = \text{diag}(r_1^*, r_2^*, ..., r_w^*)$ and $\mathbf{r}^*$ given in (4.2),

$$\mathbf{C}_0 = \mathbf{I}_{st} \otimes (\mathbf{r}^*)^\delta - \frac{1}{bstk_3}\mathbf{J}_{st} \otimes \mathbf{r}^*(\mathbf{r}^*)', \qquad \text{r}(\mathbf{C}_0) = v - 1$$

$$\mathbf{C}_1 = \mathbf{I}_{st} \otimes (\mathbf{r}^*)^\delta - \frac{1}{stk_3}\mathbf{J}_{st} \otimes \mathbf{N}_{d^*}\mathbf{N}'_{d^*}, \qquad \text{r}(\mathbf{C}_1) \le v - 1 \tag{4.4}$$

$$\mathbf{C}_2 = \mathbf{I}_s \otimes \left[\mathbf{I}_t \otimes (\mathbf{r}^*)^\delta - \frac{1}{tk_3}\mathbf{J}_t \otimes \mathbf{N}_{d^*}\mathbf{N}'_{d^*}\right], \quad \text{r}(\mathbf{C}_2) \le v - s$$

$$\mathbf{C}_3 = \mathbf{I}_{st} \otimes \left[(\mathbf{r}^*)^\delta - \frac{1}{k_3}\mathbf{N}_{d^*}\mathbf{N}'_{d^*}\right], \qquad \text{r}(\mathbf{C}_3) \le v - st.$$

It is easy to check that resulting SSP design is generally balanced. It follows from that fact the matrices (4.3) with (4.4) fulfill the condition (3.4), i.e. they commute with respect to $\mathbf{r}^{-\delta} = \mathbf{I}_{st} \otimes (\mathbf{r}^*)^{-\delta}$. Finally one can find a set of the contrasts and corresponding to them stratum efficiency factors (cf. Mejza, 1997a). We consider the following types of the contrasts: among main effects of the whole plot treatments (*A*), the subplot treatments (*B*) and the sub–subplot (*C*) treatments, including: test C treatments (C^T) and additional (control) C treatments (C^C), between the test group and the control group of C treatments (C^T *vs.* C^C), and other interaction contrasts as in table 1.

Analyzing algebraic properties of the matrices (4.3)–(4.4) we obtain information about estimability of the contrasts in the strata and their stratum efficiency factors ε_{fh} (cf. Section 3). The ε_{fh}, $h < v$; $f = 1, 2, 3, 4$, in table 1 are expressed by the eigenvalues (4.2), according to the construction method.

Table 1. Stratum efficiency factors of the considered non–orthogonal SSP design

Types of contrasts	Df	Strata			
		1	**2**	**3**	**4**
A	$s-1$		1		
B	$t-1$			1	
C^T	$\left.\begin{array}{l}\rho_1^* \\ \cdots \\ \rho_m^*\end{array}\right\} = w_1 - 1$	$1-\varepsilon_1^*$ $\cdots$ $1-\varepsilon_m^*$			ε_1^* $\cdots$ ε_m^*
C^C	$w_2 - 1$				$\varepsilon_0^* = 1$
C^T *vs* C^C	1				$\varepsilon_0^* = 1$
$A \times B$	$(s-1)(t-1)$				1
$A \times C^T$	$\left.\begin{array}{l}(s-1)\rho_1^* \\ \cdots \\ (s-1)\rho_m^*\end{array}\right\} = (s-1)(w_1 - 1)$		$1-\varepsilon_1^*$ $\cdots$ $1-\varepsilon_m^*$		ε_1^* $\cdots$ ε_m^*
$A \times C^C$	$(s-1)(w_2 - 1)$				$\varepsilon_0^* = 1$
$A \times \left(C^T \text{ } vs \text{ } C^C\right)$	$s-1$				$\varepsilon_0^* = 1$
$B \times C^T$	$\left.\begin{array}{l}(t-1)\rho_1^* \\ \cdots \\ (t-1)\rho_m^*\end{array}\right\} = (t-1)(w_1 - 1)$			$1-\varepsilon_1^*$ $\cdots$ $1-\varepsilon_m^*$	ε_1^* $\cdots$ ε_m^*

Types of contrasts	Df	Strata			
		1	2	3	4
$B\times C^C$	$(t-1)(w_2-1)$				$\varepsilon_0^*=1$
$B\times\left(C^T \textit{ vs } C^C\right)$	$t-1$				$\varepsilon_0^*=1$
$A\times B\times C^T$	$\left.\begin{matrix}(s-1)(t-1)\rho_1^*\\ \cdots\\ (s-1)(t-1)\rho_m^*\end{matrix}\right\}=$ $=(s-1)(t-1)(w_1-1)$			$1-\varepsilon_1^*$ $\cdots$ $1-\varepsilon_m^*$	ε_1^* $\cdots$ ε_m^*
$A\times B\times C^C$	$(s-1)(t-1)(w_2-1)$				$\varepsilon_0^*=1$
$A\times B\times\left(C^T \textit{ vs } C^C\right)$	$(s-1)(t-1)$				$\varepsilon_0^*=1$

Df (*degrees of freedom*) – numbers of the particular types of the contrasts estimable in the strata;
1 – the inter–block stratum, 2 – the inter–whole plot stratum, 3 – the inter–subplot stratum,
4 – the inter–sub–subplot stratum

5. Some remarks

In conclusion it can be seen that in the generated SSP design a part of the basic contrasts is estimated with full efficiency (= 1). This stratum orthogonality of the design is due to two facts. One of them is connected with the construction method, i.e. for the comparisons among main effects of the whole plot treatments (*A*), among main effects of the subplot treatments (*B*) and the interaction contrasts (*A* × *B*). The second kind of the stratum orthogonality is linked with the generating design used (orthogonally supplemented block design), i.e. for the comparisons among main effects of the control sub–subplot treatments (C^C), the interaction contrasts such as $C^T\ vs.\ C^C$, $A\times C^C$, $A\times(C^T vs.\ C^C)$, $B\times C^C$, $B\times(C^T vs.\ C^C)$, $A\times B\times C^C$, $A\times B\times(C^T vs.\ C^C)$. Other contrasts are estimated with not full efficiency in two different strata. It is worth noting that number of efficiency classes of the subdesign $\tilde{d}_1$ (and the generated SSP design also) can be reduced when we choose the PEB design with *m*–efficiency classes from the class of the PBIB designs (Mejza 1997a, 1997b). In the statistical inference about those contrasts we can use information about them separately from one stratum only or performing for them the combined estimation and testing based on information from these strata in which they are estimable (e.g. Caliński and Kageyama, 2000, Sections 3.7–3.8, 5.5). Some combining methods of information from two strata are described in Ambroży and Mejza (2006, Sections 4.4, 5.4) also.

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References

Ambroży K., Mejza I. (2006). *Doświadczenia trójczynnikowe z krzyżową i zagnieżdżoną strukturą poziomów czynników*. Wyd. Polskie Towarzystwo Biometryczne and PRODRUK, Poznań (in Polish).

Caliński T. (1971). On some desirable patterns in block designs. *Biometrics* 27, 275–292.

Caliński T., Ceranka B. (1974). Supplemented block designs. *Biom. J.* 16, 299–305.

Caliński, T., Kageyama, S. (2000). *Block Designs. A Randomization Approach*, Volume I. *Analysis*. Lecture Notes in Statistics 150, Springer–Verlag, New York.

Caliński, T., Kageyama, S. (2003). *Block Designs. A Randomization Approach*, Volume II. *Design*. Lecture Notes in Statistics 170, Springer–Verlag, New York.

Gomez K.A, Gomez A.A. (1984). *Statistical procedures for agricultural research*. Wiley, New York.

Houtman A.M., Speed T.P. (1983). Balance in designed experiments with orthogonal block structure. *Ann. Statist*. 11, 1069–1085.

Kachlicka D., Mejza I. (1998). Supplemented block designs with split units. *Colloquium Biometryczne* 28, 77–90.

Kageyama S., Puri P.D. (1985). A new special class of PEB designs. *Commun. Statist. Part A–Theor. Meth*. 14, 1731–1744.

Mejza I. (1997a). Doświadczenia trójczynnikowe w niekompletnych układach o poletkach podwójnie rozszczepionych. *Colloquium Biometryczne* 27, 127–138.

Mejza I. (1997b). Incomplete split–split–plot designs. *Abstracts of the 51st Session of the International Statistical Institute ISI'97*, Contributed Papers, Book 1, 221–222.

Mejza S. (1992). On some aspects of general balance in designed experiments. *Statistica* 52, 263–278.

Nelder J.A. (1965a). The analysis of randomized experiments with orthogonal block structure. 1. Block structure and the null analysis of variance. *Proc. Roy. Soc. London Ser. A*, 283, 147–162.

Nelder J.A. (1965b). The analysis of randomized experiments with orthogonal block structure. 2. Treatment structure and general analysis of variance. Proc. *Roy. Soc. London Ser. A*, 283, 163–178.

Nigam A.K., Puri P.D. (1982). On partially efficiency balanced designs–II. *Commun. Statist.– Theor. Meth*. 11(24), 2817–2830.

Pearce S.,C., Caliński T. and Marshall T.F. de C. (1974). The basic contrasts of an experimental design with special reference to the analysis of data. *Biometrika* 54, 449–460.

Puri P.D., Nigam A.K. (1977). Partially efficiency balanced designs. *Commun. Statist.– Theor. Meth.* 6, 753–771.

Puri P.D., Nigam A.K., Narain P. (1977). Supplemented block designs. *Sankhyā* Ser. B, 39, 189–195.

Singh M., Dey A. (1979). On analysis of some augmented block designs. *Biom. J.* 21, 87–92.