Colloquium Biometricum 41
2011, 201–210

A REMARK ON UPOV APPROACH TO TESTING UNIFORMITY OF SELF–FERTILIZED SPECIES

Wojciech Zieliński

Department of Econometrics and Statistics
Warsaw University of Live Sciences
Nowoursynowska 159, PL–02–787 Warszawa, Poland
e–mail: wojtek.zielinski@statystyka.info

Summary

The problem of testing hypothesis in a Binomial model is considered. A randomized test is constructed. An application to the problem of testing homogeneity of self–fertilized and vegetatively propagated species is shown.

Keywords and phrases: UPOV, binomial model, randomized tests

Classification AMS 2010: 62F03, 62P10

1. Introduction

One of the aspects of the investigations provided by UPOV (International Union for Protection of New Varieties of Plants) is testing of homogeneity of self–fertilized and vegetatively propagated varieties. An investigated cultivar is called uniform if the percentage of so called “off–types” is smaller than the given standard. To decide whether the variety is uniform the sample of size *n* is taken and the number of off–types is counted. To be more formal, let ξ be a number of off–types in *n* trials. This is a random variable binomially distributed. The statistical model for ξ is

$$\left(\{0,1,\ldots,n\}, \{Bin(n,\pi), \pi \in (0,1)\} \right),$$

where π denotes the population percentage of off–types. The problem is in testing hypothesis

$$H:\pi\le\pi_0 \quad \text{vs} \quad K:\pi>\pi_0$$

where π_0 is a given number (population standard). The most powerful test has the critical region

$$\{\xi>k\},$$

where k is chosen in such a way, that the test is of size α:

$$\sup_{\pi\le\pi_0} P_\pi\{\xi>k\}=P_{\pi_0}\{\xi>k\}\le\alpha\,.$$

The power of that test at the given point $\pi_1>\pi_0$ is

$$P_{\pi_1}\{\xi>k\}.$$

We are interested in finding a test of given size not greater than α and the power at least β. Such a test may be found by solving with respect to n and k a system of inequalities:

$$\begin{cases}\text{level}\,(n,k;\pi_0)\le\alpha,\\ \text{power}\,(n,k;\pi_1)\ge\beta,\end{cases}$$

where $\text{level}\,(n,k;\pi_0)=P_{\pi_0}\{\xi>k\}$ and $\text{power}\,(n,k;\pi_1)=P_{\pi_1}\{\xi>k\}$.

In UPOV documents there are given nomograms as well as tables for determining n and k for given π_0, π_1, α and β. An exemplary nomogram, similar to given in documents, is shown in the picture for $\pi_0=0.05$, $\pi_1=0.10$, $\alpha=0.05$ i $\beta=0.95$. From the picture it is found

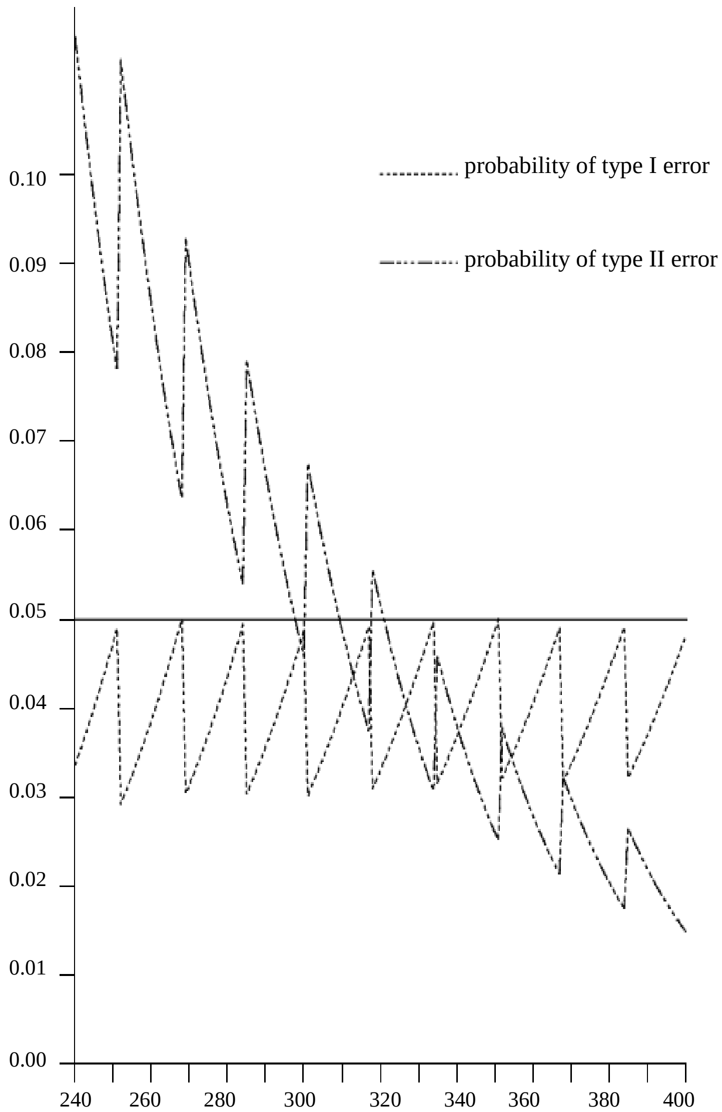
$$n=298,\qquad k=21.$$

The size as well as the power of the test equal

$$\text{level}\,(n,k\,;0.05)=0.0457643 \quad\text{and}\quad \text{power}\,(n,k\,;0.10)=0.9505957.$$

Note that the size of the test is smaller than the nominal significance level. The question is: does there exist the test with the size equal exactly given α and the power equal β? The answer is positive, but randomization is needed. In what follows the construction of randomized test is shown.

n	k
10 – 16	2
17 – 28	3
29 – 40	4
41 – 53	5
54 – 67	6
68 – 81	7
82 – 95	8
96–110	9
111–125	10
126–140	11
141–155	12
156–171	13
172–187	14
188–203	15
204–219	16
220–235	17
236–251	18
252–268	19
269–284	20
285–300	21
301–317	22
318–334	23
335–351	24
352–367	25
368–384	26
385–395	27



2. Randomized test

Recall that the cumulative distribution function of Binomial distribution may be expressed with the aid of cumulative distribution function of a Beta distribution:

$$P_\pi\{\xi \le k\} = \sum_{x=0}^{k} \binom{n}{x} \pi^x (1-\pi)^{n-x} = B(n-k, k+1; 1-\pi),$$

where $B(a,b;\cdot)$ denotes the cdf of Beta distribution with parameters a, b. Hence

$$\begin{cases} \text{level}\,(n,k;\pi_0) = 1 - B(n-k, k+1; 1-\pi_0), \\ \text{power}\,(n,k;\pi_1) = 1 - B(n-k, k+1; 1-\pi_1). \end{cases}$$

To find the appropriate test one has to solve the system of equations

$$\begin{cases} \text{level}\,(n,k;\pi_0) = \alpha \\ \text{power}\,(n,k;\pi_1) = \beta\,. \end{cases}$$

Denote the solution by n_* and k_*. Generally, those numbers are not integers. Let

$$n^* = \lfloor n_* \rfloor \text{ and } k^* = \lfloor k_* \rfloor.$$

Consider four tests with sizes and powers given below

	k^*	k^*+1
n^*	α_1	α_2
n^*+1	α_3	α_4

	k^*	k^*+1
n^*	β_1	β_2
n^*+1	β_3	β_4

where

$$\alpha_1 = \text{level}\left(n^*, k^*; \pi_0\right), \quad \alpha_2 = \text{level}\left(n^*, k^*+1; \pi_0\right),$$

$$\alpha_3 = \text{level}\left(n^*+1, k^*; \pi_0\right), \quad \alpha_4 = \text{level}\left(n^*+1, k^*+1; \pi_0\right),$$

and

$$\beta_1 = \text{power}\left(n^*, k^*; \pi_1\right), \quad \beta_2 = \text{power}\left(n^*, k^*+1; \pi_1\right),$$

$$\beta_3 = \text{power}\left(n^*+1, k^*; \pi_1\right), \quad \beta_4 = \text{power}\left(n^*+1, k^*+1; \pi_1\right).$$

Because Beta distributions are stochastically ordered so

$$\alpha_2 \le \alpha_4 \le \alpha \le \alpha_1 \le \alpha_3$$

and

$$\beta_2 \le \beta_4 \le \beta \le \beta_1 \le \beta_3\,.$$

The test is chosen by random with probabilities

	k^*	k^*+1
n^*	γ_1	γ_2
n^*+1	γ_3	γ_4

such that

$$\sum_{i=1}^{4}\alpha_i\gamma_i = \alpha \text{ and } \sum_{i=1}^{4}\beta_i\gamma_i = \beta .$$

Of course $\gamma_1+\gamma_2+\gamma_3+\gamma_4=1$.

The problem is in finding appropriate probabilities γ.

Probabilities γ are the solution of the linear system:

$$\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\\ 1 & 1 & 1 & 1\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \gamma_4\end{bmatrix}=\begin{bmatrix}\alpha\\ \beta\\ 1\end{bmatrix}.$$

The system has infinite number of solutions. Let us choose γ_4 as a free parameter. Then we obtain the following system of equations:

$$\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ 1 & 1 & 1\end{bmatrix}\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix}=\begin{bmatrix}\alpha-\alpha_4\gamma_4\\ \beta-\beta_4\gamma_4\\ 1-\gamma_4\end{bmatrix}.$$

The solution of the system is:

$$\gamma_1 = w_1 + z_1\gamma_4$$
$$\gamma_2 = w_2 + z_2\gamma_4$$
$$\gamma_3 = w_3 + z_3\gamma_4$$

where

$$w_1 = \frac{\alpha_2\beta_3-\alpha_3\beta_2+\alpha(\beta_2-\beta_3)+\beta(\alpha_3-\alpha_2)}{W},$$

$$z_1 = \frac{\alpha_2(\beta_4-\beta_3)+\alpha_3(\beta_2-\beta_4)+\alpha_4(\beta_3-\beta_2)}{W},$$

$$w_2 = \frac{\alpha_3\beta_1-\alpha_1\beta_3+\alpha(\beta_3-\beta_1)+\beta(\alpha_1-\alpha_3)}{W},$$

$$z_2 = \frac{\alpha_1(\beta_3 - \beta_4) + \alpha_3(\beta_4 - \beta_1) + \alpha_4(\beta_1 - \beta_3)}{W},$$

$$w_3 = \frac{\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha(\beta_1 - \beta_2) + \beta(\alpha_2 - \alpha_1)}{W},$$

$$z_3 = \frac{\alpha_1(\beta_4 - \beta_2) + \alpha_2(\beta_1 - \beta_4) + \alpha_4(\beta_4 - \beta_1)}{W},$$

$$W = \alpha_3(\beta_1 - \beta_2) + \alpha_1(\beta_2 - \beta_3) + \alpha_2(\beta_3 - \beta_1).$$

Solutions should be such that such that $0 \le \gamma_i \le 1$ for $i = 1, 2, 3, 4$. It gives conditions:

$$\min\{-z_i, 0\} < w_i < \max\{1, 1 - z_i\}, \ i = 1, 2, 3.$$

However, for given α and β those conditions may not be satisfied simultaneously. So requirements for the significance level and/or the power must be relaxed. It is not difficult to see that the power β must be a number such that the inequalities below holds

$$\min\{-z_1, 0\} - \frac{\alpha_2\beta_3 - \alpha_3\beta_2 + \alpha(\beta_2 - \beta_3)}{W} < \beta\frac{(\alpha_3 - \alpha_2)}{W}$$
$$< \max\{1, 1 - z_1\} - \frac{\alpha_2\beta_3 - \alpha_3\beta_2 + \alpha(\beta_2 - \beta_3)}{W},$$

$$\min\{-z_2, 0\} - \frac{\alpha_3\beta_1 - \alpha_1\beta_3 + \alpha(\beta_3 - \beta_1)}{W} < \beta\frac{(\alpha_1 - \alpha_3)}{W}$$
$$< \max\{1, 1 - z_2\} - \frac{\alpha_3\beta_1 - \alpha_1\beta_3 + \alpha(\beta_3 - \beta_1)}{W},$$

$$\min\{-z_3, 0\} - \frac{\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha(\beta_1 - \beta_2)}{W} < \beta\frac{(\alpha_2 - \alpha_1)}{W}$$
$$< \max\{1, 1 - z_3\} - \frac{\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha(\beta_1 - \beta_2)}{W}.$$

Let $\beta_{\max}$ denotes the maximal power which satisfies all above inequalities and w_1^*, w_2^* and w_3^* denote the values of w_1, w_2 and w_3 for $\beta_{\max}$, respectively. Solving inequalities

$$0 < w_1^* + z_1\gamma_4 < 1$$
$$0 < w_2^* + z_2\gamma_4 < 1$$
$$0 < w_3^* + z_3\gamma_4 < 1$$

with respect to γ_4 we obtain an interval $[\underline{\gamma}_4, \overline{\gamma}_4]$ of all admissible values of the fourth probability for which there exists the randomized test. Any of such test has the size α and the power $\beta_{\max}$.

3. Numerical example

Consider the numerical example in which

$$\pi_0 = 0.05;\ \pi_1 = 0.10;\ \alpha = 0.05;\ \beta = 0.95.$$

Values of n_* and k_* which are solutions of

$$\begin{cases} \text{level}\,(n,k;0.05) = 0.05, \\ \text{power}\,(n,k;0.10) = 0.95 \end{cases}$$

are as follows

$$n_* = 289.738, \quad k_* = 20.3203.$$

So $n^* = 289, \quad k^* = 20.$ We obtain

	20	21
289	$\alpha_1 = 0.05722$	$\alpha_2 = 0.03455$
290	$\alpha_3 = 0.05890$	$\alpha_4 = 0.03568$

	20	21
289	$\beta_1 = 0.95548$	$\beta_2 = 0.93126$
290	$\beta_3 = 0.95719$	$\beta_4 = 0.93368$

To find the randomized test one has to solve the equation:

$$\begin{bmatrix} 0.05722 & 0.03455 & 0.05890 \\ 0.95548 & 0.93126 & 0.95719 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \begin{bmatrix} 0.05 - 0.03568\,\gamma_4 \\ 0.95 - 0.93368\,\gamma_4 \\ 1 - \gamma_4 \end{bmatrix}.$$

After some calculations, the solution is obtained:

$$\gamma_1 = 26.01422 + 13.80952\gamma_4,$$
$$\gamma_2 = -1.43061 + 1.89175 \cdot 10^{-10}\gamma_4,$$
$$\gamma_3 = -23.58362 + 12.80952\gamma_4.$$

It easily seen that the condition

$$\min\{-z_1, 0\} < w_1 < \max\{1, 1 - z_1\}$$

is not fulfilled, so there does not exist required test. To find the randomized test the requirement with respect to β must be relaxed. Probabilities $\gamma_1, \gamma_2, \gamma_3$ may be written in the following way:

$$\gamma_1 = -10767.65335 + 11361.75534\,\beta - 13.80952\,\gamma_4,$$
$$\gamma_2 = 743.79752 - 784.45066\,\beta + 1.89175 \cdot 10^{-10}\,\gamma_4,$$
$$\gamma_3 = 10024.85583 - 10577.30468\,\beta + 12.80952\,\gamma_4.$$

Because γ_i for $i = 1, 2, 3$ must take on the values in the interval $[0, 1]$, we obtain

$$779.79904 - 822.7478\beta < \gamma_4 < 779.72662 - 822.7478\beta$$
$$3.9318 \cdot 10^{12} - 4.1467 \cdot 10^{12}\,\beta < \gamma_4 < 3.92651 \cdot 10^{12} - 4.1467 \cdot 10^{12}\,\beta$$
$$782.60956 - 825.73754\,\beta < \gamma_4 < 782.5315 - 825.73754\,\beta.$$

After some calculations, we obtain that the maximal β for which all above inequalities hold simultaneously and $\gamma_4 \in [0, 1]$ equals

$$\beta_{\max} = \frac{743.79752}{784.45067} = 0.94818\,.$$

The interval of admissible values of γ_4 is $[0.33520, 0.38334]$.
Any γ_4 from the above interval gives the randomized test at the level $\alpha = 0.05$ and of the power $\beta = 0.94818$. Below there are given two exemplary tests for probability γ_4 equal to the left and the right end of the interval, respectively.

$Test_L$

	$k = 20$	$k = 21$
$n = 289$	0.66480	0
$n = 290$	0	0.33520

$Test_R$

	$k = 20$	$k = 21$
$n = 289$	0	0
$n = 290$	0.61666	0.38334

Suppose that the researcher decide to apply the $Test_L$. A random number u from uniform (0,1) distribution is drawn. Then if $u < 0.66480$, then the test with $n = 289$ and $k = 20$ is used, elsewhere the test with $n = 290$ and $k = 21$ is applied.

Note that the sample size *n* in randomized test is smaller than the one in the nonrandomized test.

4. Exemplary norms

In UPOV documents there are given tests for population standards π_0=0.05, 0.03, 0.02, 0.01, 0.005, 0.001 and $\pi_1 = 2\pi_0, 5\pi_0, 10\pi_0$. Considered nominal significance levels are α=0.1, 0.05 0.01.

In the Table 1 randomized tests are presented for $\alpha = 0.05$ and power $\beta = 0.95$ (column *randomized*) for different population standards (π_0) and alternatives $\pi = 2\pi_0$. Along with those tests their exact power is calculated. Symbol

$$\begin{cases} (290, 20), & 0.61666 \\ (290, 21), & 0.38334 \end{cases}$$

means that with the probability 0.61666 the test with $n = 290$ and $k = 20$ is chosen and with the probability 0.38334 the test with $n = 290$ and $k = 21$ is applied.

In the column *UPOV* there are presented nonrandomized tests along with their actual size and power. Note that, in the randomized tests sample sizes are smaller than in the appropriate nonrandomized tests.

Table 1. Exemplary test

π_0	π_1	*randomized*	*power*	*UPOV*	*size*	*power*
0.05	0.10	$\begin{cases}(290, 20), & 0.61666\\(290, 21), & 0.38334\end{cases}$	0.94818	(298, 21)	0.04576	0.95060
0.03	0.06	$\begin{cases}(499, 20), & 0.01619\\(499, 21), & 0.98381\end{cases}$	0.94994	(519, 22)	0.04343	0.95020
0.02	0.04	$\begin{cases}(760, 21), & 0.30062\\(761, 21), & 0.34334\\(761, 22), & 0.35604\end{cases}$	0.94844	(839, 24)	0.03439	0.95007
0.01	0.02	$\begin{cases}(1545, 21), & 0.32016\\(1545, 22), & 0.67984\end{cases}$	0.94848	(1625, 23)	0.04156	0.95019
0.005	0.01	$\begin{cases}(3111, 21), & 0.14906\\(3112, 21), & 0.03349\\(3112, 22), & 0.81745\end{cases}$	0.94891	(3254, 23)	0.04246	0.95005
0.001	0.002	$\begin{cases}(15655, 21), & 0.06856\\(15655, 22), & 0.93144\end{cases}$	0.94954	(16288, 23)	0.04321	0.95000

References

Laidig F. (1989). Testing of homogeneity of self–fertilized and vegetatively propagated species, maximum number of off–types and parameters for sampling. *TWC* VII/4, UPOV Geneva.

Kristensen K. (1997). Testing of homogeneity of self–fertilized and vegetatively propagated species using off–types. *TWC* 15/12, UPOV Geneva.

Examining uniformity. (1989). Document TGP/10, Geneva.