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ON SOME D-OPTIMAL CHEMICAL BALANCE WEIGHING DESIGN WITH $n \equiv 2 \pmod 4$

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Summary

In this paper, we consider the chemical balance weighing designs for estimation of individual unknown weights of three objects using D-optimality criterion. We assume that the error components create a first-order autoregressive process AR(1). Then, the covariance matrix of random errors has known form, which does not have to be identity matrix and depends on known parameter ρ. In this paper, we prove D-optimality of some design from Bora-Senta and Moyssiadis (1999), if $n \equiv 2 \pmod 4$, in the whole class of designs for three objects and some $\rho \le 0$. Under these assumptions, we present the necessary and sufficient conditions such that the weighing design for three objects is D-optimal. These conditions can be used to construct D-optimal designs.

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1. Introduction

In the paper we consider the chemical balance, where each object can be placed on one of two pans (left and right). A reading represents the total weight of the objects on the pans. We would like to choose a chemical balance weighing design that is optimal with respect to D-optimality criterion, which we define below.

At the beginning, we introduce a model for chemical balance weighing design for three objects. We estimate the true unknown weights $\omega_1, \omega_2, \omega_3$ of three objects employing n measuring operations using a chemical balance. Let $y_1, y_2, \ldots, y_n$ denote the observations in these n operations, respectively. We assume that the observations follow the linear model $\mathbf{y} = \mathbf{X}\boldsymbol{\omega} + \boldsymbol{\varepsilon}$, where $\mathbf{y} = [y_1, y_2, \ldots, y_n]'$ is an $n \times 1$ vector of observations, $\boldsymbol{\omega} = [\omega_1, \omega_2, \omega_3]'$ is the vector of unknown weights of objects, the $n \times 3$ matrix $\mathbf{X} = [x_{ij}]$ is called the design matrix, the vector $\boldsymbol{\varepsilon} = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]'$ is the vector of error components. In the chemical balance weighing design, we suppose that $x_{ij} = -1$ $(x_{ij} = 1)$ if the *j*th object is placed on the left (right) pan during the *i*th weighing operation. We consider the case when the random errors form an AR(1) process which implies that $E(\boldsymbol{\varepsilon}) = [0, 0, \ldots, 0]'$ is an $n \times 1$ nil vector and $Var(\boldsymbol{\varepsilon}) = 1/(1-\rho^2)\mathbf{S}$, where $\mathbf{S} = (\rho^{|r-d|})_{r,d=1}^{n}$ and $-1 < \rho < 1$. We identify the design with its matrix $\mathbf{X}$.

The D-optimal chemical balance weighing design maximizes the determinant of the information matrix $\mathbf{X}'\mathbf{S}^{-1}\mathbf{X}$. More precisely, the design $\tilde{\mathbf{X}}$ is D-optimal in the class of the designs $C \subseteq M_{n\times 3}(\pm 1)$, where the set $M_{n\times p}(\pm 1)$ consists of all matrices with n rows, p columns and elements 1 or -1, if $\det(\tilde{\mathbf{X}}'\mathbf{S}^{-1}\tilde{\mathbf{X}}) = \max\{\det(\mathbf{X}'\mathbf{S}^{-1}\mathbf{X}) : \mathbf{X} \in C\}$.

The case, when the matrix $\mathbf{S}$ is the identity matrix $(\rho = 0)$, is well known and the D-optimal designs are considered in many papers (see, e.g. Galil and Kiefer (1980), or Jacroux et al. (1983)). For $\rho \neq 0$, Bora-Senta and Moyssiadis (1999) gave some conjectures (based on several exhaustive searches) about D-optimal chemical balance weighing designs with matrices $\mathbf{X} = [\mathbf{1}_n \mid \mathbf{x} \mid \mathbf{y}] \in M_{n\times 3}(\pm 1)$, where $\mathbf{1}_n$ is the vector of n ones. These conjectures were proved in Li and Yang (2005) and Yeh and Lo Huang (2005) for $n \equiv 0 \pmod 4$, $\rho \in (-1,1)$ and $n \equiv 2 \pmod 4$, $\rho > 0$. For some $-1 < \rho \leq 0$ and $n \equiv 0 \pmod 4$, some construction of D-optimal design in the class of designs such that each column of the design matrix $\mathbf{X}$ contains at least one 1 and one -1 were considered in Katulska and Smaga (2010) and Katulska and Smaga (accepted).

Some results about D-optimal designs in the classes of designs with matrices $\mathbf{X} = [\mathbf{x} \mid \mathbf{y} \mid \mathbf{z}] \in M_{n\times 3}(\pm 1)$ and $\mathbf{X} = [\mathbf{1}_n \mid \mathbf{x} \mid \mathbf{y} \mid \mathbf{z}] \in M_{n\times 4}(\pm 1)$ for some $\rho \ge 0$ are given in Katulska and Smaga (2012) and Katulska and Smaga (2011), respectively.

In Theorem 2.5 of paper, we prove the conjecture from Bora-Senta and Moyssiadis (1999), if $n \equiv 2 \pmod 4$ and some $\rho \le 0$ in the class $M_{n\times 3}(\pm 1)$. The necessary and sufficient conditions under which the design is D-optimal in the class of designs with these assumptions are also given.

2. D-optimal chemical balance weighing designs

In this section, we present the main results but first we give some definitions and supporting results.

For any vector $\mathbf{x} = [x_1, x_2, \dots, x_n]' \in M_{n\times 1}(\pm 1)$, we define the numbers

$$cs(\mathbf{x}) = \#\{i : x_i = -x_{i+1}, 1 \le i \le n-1\},$$

$$fcs(\mathbf{x}) = \min\{i : x_i = -x_{i+1}, 1 \le i \le n-1\},$$

$$scs(\mathbf{x}) = \min\{i : i > fcs(\mathbf{x}), x_i = -x_{i+1}, 1 \le i \le n-1\}.$$

We obtain the following lemma directly from properties of determinants (see Horn and Johnson, 1985).

Lemma 2.1. If $\mathbf{X} \in M_{n\times p}(\pm 1)$ and $\mathbf{G}$ is the $n \times n$ real matrix, then the determinant of the matrix $\mathbf{X}'\mathbf{G}\mathbf{X}$ does not change if we interchange two columns of the matrix $\mathbf{X}$ or we multiply any column of this matrix by -1.

Below, we remind well known inequality.

Lemma 2.2. (Fischer's inequality). If $\mathbf{P} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{D} \end{bmatrix}$ is a positive definite matrix that is partitioned so that $\mathbf{B}$ and $\mathbf{D}$ are square and nonempty, then $\det(\mathbf{P}) \le \det(\mathbf{B})\det(\mathbf{D})$ and the equality holds if and only if $\mathbf{C} = \mathbf{0}$.

Lemma 2.3. Suppose that $n \equiv 2 \pmod 4$ and $\lambda = 0, 1, 2, \dots, n-1$. If $\Delta = (n-2)(1-\rho)^2 + 2(1-\rho)$, $\rho \ne 0$ and $\mathbf{x} \in M_{n\times 1}(\pm 1)$, then $cs(\mathbf{x}) = \lambda$ if and only if $\mathbf{x}'\mathbf{A}\mathbf{x} = \Delta + 4\lambda\rho$, where

$$\mathbf{A}=\begin{bmatrix} 1 & -\rho & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}. \qquad (2.1)$$

Proof. The thesis follows from equality

$$\mathbf{x}'\mathbf{A}\mathbf{x}=(n-2)(1+\rho^2)+2-2\rho(x_1x_2+x_2x_3+\cdots+x_{n-1}x_n). \ \blacksquare$$

The next lemma follows from proofs in Yeh and Lo Huang (2005) and some direct calculations.

Lemma 2.4. Let

$\mathbf{x}=[x_1, x_2, \ldots, x_n]'$, $\mathbf{y}=[y_1, y_2, \ldots, y_n]' \in M_{n\times 1}(\pm 1)$, $n \equiv 2 \pmod 4$ and the matrix $\mathbf{A}$ is defined by (2.1).

(a) If $cs(\mathbf{x})=cs(\mathbf{y})=1$, $fcs(\mathbf{x})>fcs(\mathbf{y})$, then

$$\mathbf{x}'\mathbf{A}\mathbf{y}=\begin{cases} (n-2fcs(\mathbf{x})+2fcs(\mathbf{y})-2)(1-\rho)^2+2(1-\rho) & if\ x_1=y_1 \\ -((n-2fcs(\mathbf{x})+2fcs(\mathbf{y})-2)(1-\rho)^2+2(1-\rho)) & if\ x_1\neq y_1 \end{cases}.$$

(b) If $cs(\mathbf{x})=0$, $cs(\mathbf{y})=2$, then

$$\mathbf{x}'\mathbf{A}\mathbf{y}=\begin{cases} (n+2fcs(\mathbf{y})-2scs(\mathbf{y})-2)(1-\rho)^2+2(1-\rho) & if\ x_1=y_1 \\ -((n+2fcs(\mathbf{y})-2scs(\mathbf{y})-2)(1-\rho)^2+2(1-\rho)) & if\ x_1\neq y_1 \end{cases}.$$

(c) If $cs(\mathbf{x})=0$, $cs(\mathbf{y})=1$, then

$$\mathbf{x}'\mathbf{A}\mathbf{y}=\begin{cases} (2fcs(\mathbf{y})-n)(1-\rho)^2 & if\ x_1=y_1 \\ -(2fcs(\mathbf{y})-n)(1-\rho)^2 & if\ x_1\neq y_1 \end{cases}.$$

(d) If $cs(\mathbf{x})=1$, $fcs(\mathbf{x})=n/2$, $cs(\mathbf{y})=2$, $b=fcs(\mathbf{y})$, $c=scs(\mathbf{y})$, then

$$\mathbf{x}'\mathbf{A}\mathbf{y} = \begin{cases} 2(b+c-n)(1-\rho)^2 & \text{if } x_1 = y_1, b < n/2, c > n/2 \\ -2(b+c-n)(1-\rho)^2 & \text{if } x_1 \neq y_1, b < n/2, c > n/2 \\ (n-4)(1-\rho)^2 + 2(1+\rho^2) & \text{if } (x_1 \neq y_1, b = 1, c = n/2) \text{ or } \\ & \quad (x_1 = y_1, b = n/2, c = n-1) \\ -[(n-4)(1-\rho)^2 + 2(1+\rho^2)] & \text{if } (x_1 = y_1, b = 1, c = n/2) \text{ or } \\ & \quad (x_1 \neq y_1, b = n/2, c = n-1) \end{cases}.$$

Now, we formulate new theorems concerning D-optimal chemical balance weighing designs under the assumption that the random errors form a process AR(1). First, we prove that some design is D-optimal weighing design for three objects and some $\rho \leq 0$.

Theorem 2.5. Let $n \equiv 2 \pmod 4$, $n \neq 2$ and $\rho \in (-1, -1/(n-2)] \cup \{0\}$ if $n = 6, 10, \ldots, 22$, and $\rho \in (-4/(n-8), -1/(n-2)] \cup \{0\}$ if $n \geq 26$. Then the design with the matrix

$$\hat{\mathbf{X}} = \begin{bmatrix} 1 & 1 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 1 & 1 \\ 1 & 1 & -1_2 \\ \vdots & \vdots & \vdots \\ 1 & 1 & -1 \\ 1 & -1_1 & -1 \\ \vdots & \vdots & \vdots \\ 1 & -1 & -1 \\ 1 & -1 & 1_3 \\ \vdots & \vdots & \vdots \\ 1 & -1 & 1 \end{bmatrix}, \tag{2.2}$$

where elements with indices 1, 2 and 3 are in positions $(n/2+1,\ 2), ((n-2)/4+2,\ 3), (3(n-2)/4+2,\ 3)$, respectively, is D-optimal chemical balance weighing design for three objects.

Proof. (Sketch) The inverse of the matrix $\mathbf{S}$ is equal to $\mathbf{S}^{-1} = 1/(1-\rho^2)\mathbf{A}$, where the matrix $\mathbf{A}$ is given by (2.1). The matrix $\mathbf{A}$ is positive definite. From definition of D-optimal design and the inverse of the matrix $\mathbf{S}$ we obtain the D-optimal design in the class of designs $C \subseteq M_{n\times p}(\pm 1)$ maximizes the determinant of the matrix $\mathbf{X}'\mathbf{AX}$ among all $\mathbf{X} \in C$.
From Lemmas 2.3 and 2.4 for the matrix $\hat{\mathbf{X}}$ of the form (2.2), we have

$$\det(\hat{\mathbf{X}}'\mathbf{A}\hat{\mathbf{X}}) = \det\begin{bmatrix} \Delta & 0 & 2(1-\rho) \\ 0 & \Delta+4\rho & 0 \\ 2(1-\rho) & 0 & \Delta+8\rho \end{bmatrix} = (\Delta+4\rho)[\Delta(\Delta+8\rho)-4(1-\rho)^2].$$

When $\rho = 0$, then the matrix $\mathbf{A}$ is the identity matrix and

$$\det(\hat{\mathbf{X}}'\mathbf{A}\hat{\mathbf{X}}) = \det(\hat{\mathbf{X}}'\hat{\mathbf{X}}) = \det\begin{bmatrix} n & 0 & 2 \\ 0 & n & 0 \\ 2 & 0 & n \end{bmatrix} = n^3 - 4n.$$

Hence $\hat{\mathbf{X}}$ is D-optimal from Jacroux et al. (1983). From now on, we assume that $\rho \neq 0$. It is easy to see that the matrix $\mathbf{X}'\mathbf{AX}$ is positive definite. By Lemma 2.1, we can suppose $x_1 = y_1 = z_1 = 1$ and consider only the designs with matrices $\mathbf{X} = [\mathbf{x} \,|\, \mathbf{y} \,|\, \mathbf{z}] \in C_1 \cup C_2 \cup C_3$, where

$$C_1 = \{[\boldsymbol{\alpha} \,|\, \boldsymbol{\beta} \,|\, \boldsymbol{\gamma}] \in M_{n\times 3}(\pm 1) : cs(\boldsymbol{\alpha}) \geq 1,\, cs(\boldsymbol{\beta}) \geq 1,\, cs(\boldsymbol{\gamma}) \geq 2\},$$

$$C_2 = \{[\boldsymbol{\alpha} \,|\, \boldsymbol{\beta} \,|\, \boldsymbol{\gamma}] \in M_{n\times 3}(\pm 1) : cs(\boldsymbol{\alpha}) = 0,\, cs(\boldsymbol{\beta}) \geq 1,\, cs(\boldsymbol{\gamma}) \geq 1\},$$

$$C_3 = \{[\boldsymbol{\alpha} \,|\, \boldsymbol{\beta} \,|\, \boldsymbol{\gamma}] \in M_{n\times 3}(\pm 1) : cs(\boldsymbol{\alpha}) = cs(\boldsymbol{\beta}) = cs(\boldsymbol{\gamma}) = 1\}.$$

We show that $\det(\hat{\mathbf{X}}'\mathbf{A}\hat{\mathbf{X}}) \geq \det(\mathbf{X}'\mathbf{AX})$ for all $\mathbf{X} \in C_i,\, i = 1, 2, 3$. For example, we present the proof if $\mathbf{X} = [\mathbf{x} \,|\, \mathbf{y} \,|\, \mathbf{z}] \in C_1$. Then from Hadamard's inequality, the determinant of the matrix $\mathbf{X}'\mathbf{AX}$ is less or equal to the product of the diagonal elements of this matrix, ie $\det(\mathbf{X}'\mathbf{AX}) \leq (\mathbf{x}'\mathbf{Ax})(\mathbf{y}'\mathbf{Ay})(\mathbf{z}'\mathbf{Az})$. From Lemma 2.3, we obtain the inequalities $\mathbf{x}'\mathbf{Ax} \leq \Delta + 4\rho$, $\mathbf{y}'\mathbf{Ay} \leq \Delta + 4\rho$, $\mathbf{z}'\mathbf{Az} \leq \Delta + 8\rho$. Therefore we conclude $\det(\mathbf{X}'\mathbf{AX}) \leq (\Delta+4\rho)^2(\Delta+8\rho)$ and

$$\det(\hat{\mathbf{X}}'\mathbf{A}\hat{\mathbf{X}}) - \det(\mathbf{X}'\mathbf{AX}) \geq \det(\hat{\mathbf{X}}'\mathbf{A}\hat{\mathbf{X}}) - (\Delta+4\rho)^2(\Delta+8\rho)$$
$$= 4(\Delta+4\rho)[-(n-2)\rho^3 + (2n-11)\rho^2 - (n-2)\rho - 1] > 0,$$

which completes the proof. ■

From the proof of Theorem 2.5, it follows that the design $\hat{\mathbf{X}}$ given by (2.2) is D-optimal in some large subclass of the class $M_{n\times 3}(\pm 1)$ for all $\rho \in (-1, -1/(n-2)] \cup \{0\}$, what we describe in the following corollary.

Corollary 2.6. If $\rho \in (-1, -1/(n-2)] \cup \{0\}$ and $n \equiv 2 \pmod 4$, $n \neq 2$, then the design $\hat{\mathbf{X}}$ given by (2.2) is D-optimal in the class

$\{[\boldsymbol{\alpha} \mid \boldsymbol{\beta} \mid \boldsymbol{\gamma}] \in M_{n\times 3}(\pm 1) : cs(\boldsymbol{\alpha}) \geq 0, cs(\boldsymbol{\beta}) \geq 1, cs(\boldsymbol{\gamma}) \geq 2 \ or\ cs(\boldsymbol{\alpha}) = cs(\boldsymbol{\beta}) = cs(\boldsymbol{\gamma}) = 1\}$.

Now, we prove some necessary and sufficient conditions under which the design for the three objects is the D-optimal.

Theorem 2.7. If n and ρ are the same as in Theorem 2.5, $\mathbf{X}^* = [\mathbf{x}^* \mid \mathbf{y}^* \mid \mathbf{z}^*] \in M_{n\times 3}(\pm 1)$, then the design $\mathbf{X}^*$ is D-optimal in the class of designs for three objects if and only if

$$\mathbf{X}^{*\prime}\mathbf{A}\mathbf{X}^* = \begin{bmatrix} \Delta & 0 & \pm 2(1-\rho) \\ 0 & \Delta + 4\rho & 0 \\ \pm 2(1-\rho) & 0 & \Delta + 8\rho \end{bmatrix} \tag{2.3}$$

exact to permuting columns of the matrix $\mathbf{X}^*$.

Proof. We present the proof if $\rho \neq 0$. First, we prove the sufficient condition. If the design $\mathbf{X}^*$ satisfies the equality (2.3), then by Theorem 2.5 we obtain $\det(\mathbf{X}^{*\prime}\mathbf{A}\mathbf{X}^*) = \det(\hat{\mathbf{X}}'\mathbf{A}\hat{\mathbf{X}})$, so the design $\mathbf{X}^*$ is D-optimal in $M_{n\times 3}(\pm 1)$. Now, we present the necessary condition. Assume that $\mathbf{X}^*$ is the D-optimal design for three objects. So by Theorem 2.5, we conclude that $\det(\mathbf{X}^{*\prime}\mathbf{A}\mathbf{X}^*) = \det(\hat{\mathbf{X}}'\mathbf{A}\hat{\mathbf{X}}) = (\Delta + 4\rho)[\Delta(\Delta + 8\rho) - 4(1-\rho)^2]$. From the proof of Theorem 2.5, we obtain $\det(\mathbf{X}^{*\prime}\mathbf{A}\mathbf{X}^*) > \det(\mathbf{X}'\mathbf{A}\mathbf{X})$ for all designs $\mathbf{X} \in M_{n\times 3}(\pm 1) \setminus B$, where

$$B = \left\{[\boldsymbol{\alpha} \mid \boldsymbol{\beta} \mid \boldsymbol{\gamma}] : cs(\boldsymbol{\alpha}) = 0, cs(\boldsymbol{\beta}) = 1, cs(\boldsymbol{\gamma}) = 2,\ fsc(\boldsymbol{\beta}) = \frac{n}{2}, scs(\boldsymbol{\gamma}) - fcs(\boldsymbol{\gamma}) \neq \frac{n}{2}\right\}.$$

If $\mathbf{X}^* \in B$, then from Lemma 2.3 it follows that $\mathbf{x}^{*\prime}\mathbf{A}\mathbf{x}^* = \Delta$, $\mathbf{y}^{*\prime}\mathbf{A}\mathbf{y}^* = \Delta + 4\rho$ and $\mathbf{z}^{*\prime}\mathbf{A}\mathbf{z}^* = \Delta + 8\rho$. By Lemma 2.1:

$$\det(\mathbf{X}^{*\prime}\mathbf{A}\mathbf{X}^*) = \det([\mathbf{x}^* \mid \mathbf{z}^* \mid \mathbf{y}^*]'\mathbf{A}[\mathbf{x}^* \mid \mathbf{z}^* \mid \mathbf{y}^*]).$$

From Fischer's inequality, we obtain the following inequality

$$\det(\mathbf{X}^{*}{}'\mathbf{A}\mathbf{X}^{*}) \le (\Delta + 4\rho)[\Delta(\Delta + 8\rho) - (\mathbf{x}^{*}{}'\mathbf{A}\mathbf{z}^{*})^{2}]. \qquad (2.4)$$

The equality in (2.4) holds if and only if $\mathbf{x}^{*}{}'\mathbf{A}\mathbf{y}^{*} = \mathbf{y}^{*}{}'\mathbf{A}\mathbf{z}^{*} = 0.$ Moreover, from the fact that $scs(\mathbf{z}^{*}) - fcs(\mathbf{z}^{*}) \ne n/2$ and Lemma 2.4 (b), it follows that $(\mathbf{x}^{*}{}'\mathbf{A}\mathbf{z}^{*})^{2} \ge 4(1-\rho)^{2}$ and the equality holds if and only if $\mathbf{x}^{*}{}'\mathbf{A}\mathbf{z}^{*} = \pm 2(1-\rho).$ Therefore, we obtain the following inequality

$$\det(\mathbf{X}^{*}{}'\mathbf{A}\mathbf{X}^{*}) \le (\Delta + 4\rho)[\Delta(\Delta + 8\rho) - 4(1-\rho)^{2}] = \det(\hat{\mathbf{X}}'\mathbf{A}\hat{\mathbf{X}}). \qquad (2.5)$$

But as we noted at the beginning of the proof in the inequality (2.5) there must be equality. So $\mathbf{x}^{*}{}'\mathbf{A}\mathbf{y}^{*} = \mathbf{y}^{*}{}'\mathbf{A}\mathbf{z}^{*} = 0, \mathbf{x}^{*}{}'\mathbf{A}\mathbf{z}^{*} = \pm 2(1-\rho)$ and the matrix $\mathbf{X}^{*}{}'\mathbf{A}\mathbf{X}^{*}$ has the form (2.3). ■

Theorem 2.8. Let $n \equiv 2 \ (\mathrm{mod}\ 4), n \ne 2$ and $\rho \in \left(-1, -1/(n-2)\right]$ if $n = 6, 10, \ldots, 22,$ and $\rho \in \left(-4/(n-8), -1/(n-2)\right]$ if $n \ge 26.$ Then the design $\mathbf{X}^{*} = [\mathbf{x}^{*} \mid \mathbf{y}^{*} \mid \mathbf{z}^{*}] \in M_{n\times 3}(\pm 1)$ is D-optimal in the class of designs for three objects if and only if $cs(\mathbf{x}^{*}) = 0,$ $cs(\mathbf{y}^{*}) = 1,$ $cs(\mathbf{z}^{*}) = 2$ and $fcs(\mathbf{y}^{*}) = n/2,$ $fcs(\mathbf{z}^{*}) = (n-2)/4 + 1,$ $scs(\mathbf{z}^{*}) = 3(n-2)/4 + 1$ exact to permuting columns of the matrix $\mathbf{X}^{*}$.

Proof. The sufficient condition is easy to see, because from Lemmas 2.3 and 2.4, we conclude that the matrix $\mathbf{X}^{*}{}'\mathbf{A}\mathbf{X}^{*}$ has the form (2.3) and hence by Theorem 2.7, the design $\mathbf{X}^{*}$ is D-optimal design for three objects. Proof of necessary condition is as follows. Let $\mathbf{X}^{*}$ be the D-optimal design for three objects. So the matrix $\mathbf{X}^{*}{}'\mathbf{A}\mathbf{X}^{*}$ has the form (2.3) by Theorem 2.7.

From Lemma 2.3, it follows that $\mathbf{x}^{*}{}'\mathbf{A}\mathbf{x}^{*} = \Delta \Leftrightarrow cs(\mathbf{x}^{*}) = 0,$ $\mathbf{y}^{*}{}'\mathbf{A}\mathbf{y}^{*} = \Delta + 4\rho \Leftrightarrow cs(\mathbf{y}^{*}) = 1$ and $\mathbf{z}^{*}{}'\mathbf{A}\mathbf{z}^{*} = \Delta + 8\rho \Leftrightarrow cs(\mathbf{z}^{*}) = 2.$ Moreover, from Lemma 2.4 (c), we have $\mathbf{x}^{*}{}'\mathbf{A}\mathbf{y}^{*} = \pm(2\,fcs(\mathbf{y}^{*}) - n)(1-\rho)^{2} = 0,$ so $fcs(\mathbf{y}^{*}) = n/2.$ From the equality $\mathbf{x}^{*}{}'\mathbf{A}\mathbf{z}^{*} = \pm 2(1-\rho)$ and Lemma 2.4 (b), we obtain $scs(\mathbf{z}^{*}) - fcs(\mathbf{z}^{*}) = n/2 - 1.$ Hence and from the fact that $\mathbf{y}^{*}{}'\mathbf{A}\mathbf{z}^{*} = 0$ we have (by Lemma 2.4 (d)) $fcs(\mathbf{z}^*) < n/2$, $scs(\mathbf{z}^*) > n/2$ and hence $\mathbf{y}^{*}{}'\mathbf{A}\mathbf{z}^* = \pm 2(fcs(\mathbf{z}^*) + scs(\mathbf{z}^*) - n)(1-\rho)^2 = 0$ which implies $fcs(\mathbf{z}^*) + scs(\mathbf{z}^*) = n.$

Therefore $fcs(\mathbf{z}^*) = (n-2)/4+1,$ $scs(\mathbf{z}^*) = 3(n-2)/4+1.$ So the thesis is proved. ■

Using Theorems 2.7 and 2.8, D-optimal chemical balance weighing designs (other than $\hat{\mathbf{X}}$) for the three objects under the assumption that the random errors form a process AR(1) can be constructed.

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