Colloquium Biometricum 44
2014, 139–148

# ON MULTIVARIATE NORMALITY TESTS USING SKEWNESS AND KURTOSIS

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## Summary

In the paper two new tests for multivariate normality are proposed. The tests are based on Mardia's and Srivastava's more accurate moments of multivariate sample skewness and kurtosis. Sample significance level and power against chosen alternative distributions of both tests were calculated via simulation studies. The obtained results have been compared to the results of two improved the Jarque-Bera's tests and the Henze-Zirkler test

**Keywords and phrases**: multivariate normality, multivariate skewness, multivariate kurtosis

**Classification AMS 2010**: 62H15, 62H10

## 1. Introduction

Testing normality of multivariate data sets is a crucial task in multivariate analysis. There are many different tests devoted to this problem (Doornik and Hansen, 2008; Farrel et al., 2007; Henze and Zirkler, 1990; Mecklin and Mundfrom, 2004; Romeu and Ozturk, 1993). Mostly, tests for multivariate normality are generalization of tests for univariate normality. One group of such tests is based on multivariate skewness and kurtosis (Mardia, 1970, 1974; Srivastava, 1984, 2002). Jarque and Bera (1987) proposed the test combining both Mardia's skewness and kurtosis. Hanusz et al. (2014) consider some improvement of the Jarque-Bera's test using more accurate moments of sample multivariate kurtosis and skewness given by Mardia (1974). They also proposed the test analogous to Jarque-Bera, based on multivariate Srivastava's skewness and kurtosis (Srivastava, 1984). In the paper we consider similar idea to construct some other tests involving more accurate moments of multivariate sample skewness and kurtosis given by Mardia and Srivastava. The null distribution of these tests is Student's t. Sample significance level and power of proposed tests are compared with the improved Jarque-Bera tests with more accurate moments and the Henze-Zirkler test. The biggest advantages of the tests under consideration are their known asymptotic distributions, invariance and consistence.

## 2. Description of compared tests for multivariate normality

Let us consider random sample $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$ of independent and identical p-variate vectors with unknown mean $\boldsymbol{\mu}$ and unknown covariance matrix $\boldsymbol{\Sigma}$. Mardia (1970, 1974) defined the measure of multivariate skewness and kurtosis as follows

$$b_{1,p}^{M} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left\{\left(\mathbf{X}_i - \overline{\mathbf{X}}\right)'\mathbf{S}^{-1}\left(\mathbf{X}_j - \overline{\mathbf{X}}\right)\right\}^3 ,$$

$$b_{2,p}^{M} = \frac{1}{n}\sum_{i=1}^{n}\left\{\left(\mathbf{X}_i - \overline{\mathbf{X}}\right)'\mathbf{S}^{-1}\left(\mathbf{X}_i - \overline{\mathbf{X}}\right)\right\}^2 ,$$

respectively, where $\overline{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i$ and $\mathbf{S} = \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{X}_i - \overline{\mathbf{X}}\right)'\left(\mathbf{X}_i - \overline{\mathbf{X}}\right)$ denote sample mean and sample covariance matrix, $\mathbf{S}^{-1}$ is the inverse of $\mathbf{S}$.

Under normality of $\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_n$, the statistic $A = \frac{n}{6}b_{1,p}^{M}$ has asymptotic chi-square distribution with $f = \frac{1}{6}p(p+1)(p+2)$ degrees of freedom and the statistic $B = \frac{b_{2,p}^{M} - p(p+2)}{\sqrt{8p(p+2)/n}}$ has asymptotic standard normal distribution (Mardia, 1970).

Based on the statistics A and B, as test for multivariate normality Jarque and Bera (1987) proposed to use the statistic $JB = A + B^2$ which has asymptotic chi-square distribution with $f+1$ degrees of freedom.

Using more accurate moments of skewness and kurtosis given in Mardia (1974) we get improved tests

$$A' = \frac{nK}{6} b_{1,p}^M \text{ where } K = \frac{(p+1)(n+1)(n+3)}{n\{(n+1)(p+1)-6\}},$$

$$B' = \frac{(n+1)b_{2,p}^M - p(p+2)(n-1)}{\sqrt{8p(p+2)(n-3)(n-p-1)(n-p+1)/\{(n+3)(n+5)\}}},$$

and improved Jarque-Bera's test

$$JB'_M = A' + (B')^2 . \tag{2.1}$$

Srivastava (1984) formulated the definition of multivariate sample skewness and kurtosis using principal components, namely,

$$b_{1,p}^S = \frac{1}{pn^2} \sum_{i=1}^{p} \frac{1}{w_i^3} \left\{ \sum_{j=1}^{n} \left( Y_{ij} - \bar{Y}_i \right)^3 \right\}^2 ,$$

$$b_{2,p}^S = \frac{1}{pn} \sum_{i=1}^{p} \frac{1}{w_i^2} \sum_{j=1}^{n} \left( Y_{ij} - \bar{Y}_i \right)^4 ,$$

where $Y_{ij} = \mathbf{u}_i' \mathbf{X}_j$, $\bar{Y}_i = \frac{1}{n} \sum_{i=1}^{n} Y_{ij}$, $\mathbf{H} = [\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_p]$ is an orthogonal matrix such that $\mathbf{S} = \mathbf{H}\mathbf{D}_w\mathbf{H}'$, $\mathbf{D}_w = diag(w_1, w_2, \cdots, w_p)$ and $w_1, w_2, \cdots, w_p$ are eigenvalues of $\mathbf{S}$ $(i = 1,2,\cdots, p; j = 1,2,\cdots n)$.

Under normality of the **X**'s, the statistic $S_1 = \frac{np}{6} b_{1,p}^S$ has asymptotic chi-square distribution with $p$ degrees of freedom and the statistic $S_2 = \frac{b_{2,p}^S - 3}{\sqrt{24/np}}$

has asymptotic standard normal distribution. If more accurate moments of skewness and kurtosis are used (Seo and Ariga, 2011; Hanusz et al., 2012; Koizumi et al., 2009) then the improved tests have the following forms

$$S_1^* = \frac{p(n+1)(n+3)}{6(n-2)} b_{1,p}^S ,$$

$$S_2^* = \frac{(n+1)b_{2,p}^S - 3(n-1)}{\sqrt{24n(n-3)(n-2)/\{p(n+3)(n+5)\}}} .$$

Additionally, $S_1^*$ and $S_2^*$ have asymptotic chi-square distribution with $p$ degrees of freedom and standard normal, respectively.

Applying the idea of Jarque and Bera the improved test has the following form

$$JB_S' = S_1^* + \left(S_2^*\right)^2 , \tag{2.2}$$

which has an asymptotic chi-square distribution with $p$+1 degrees of freedom.

Sample significance levels and sample power of the tests described by (2.1) and (2.2) are presented in Hanusz et al. (2014).

In the paper we propose two other tests of the forms

$$T_M = \frac{B'}{\sqrt{A'/f}} , \tag{2.3}$$

and

$$T_S = \frac{S_2^*}{\sqrt{S_1^*/p}} , \tag{2.4}$$

having asymptotic Student $t$ distribution with $f = \frac{1}{6} p(p+1)(p+2)$ and $p$ degrees of freedom, respectively. In the next section, sample significance level and sample power of the test $T_M$ and $T_S$ are presented. The results are compared to counterparts of the improved Jarque-Bera tests $JB_M'$, $JB_S'$ and the Henze-Zirkler invariant consistent tests for MVN (Henze and Zirkler, 1990). The test statistic has the form $T_{n,\beta} = n\left(4\mathrm{I}\{\mathbf{S}\text{ is singular}\} + D_{n,\beta}\mathrm{I}\{\mathbf{S}\text{ is nonsingular}\}\right)$, where $\mathbf{S}$ is a sample covariance matrix, $D_{n,\beta} = \int_{R^p} \left|\Psi_n(\mathbf{t}) - \exp\left(-\frac{1}{2}\mathbf{t}'\mathbf{t}\right)\right|^2 \varphi_\beta(\mathbf{t})d\mathbf{t}$, $\mathrm{I}\{\cdot\}$ is the indicator function, $\varphi_\beta(\mathbf{t}) = \left(2\pi\beta^2\right)^{-\frac{p}{2}} \exp\left(-\frac{\mathbf{t}'\mathbf{t}}{2\beta^2}\right)$, $\Psi_n(\mathbf{t}) = \frac{1}{n}\sum_{j=1}^{n} \exp\left(i\mathbf{t}'\,\mathbf{Y}_j\right)$ is the empirical characteristic function of scaled residuals $\mathbf{Y}_j = \mathbf{S}^{-\frac{1}{2}}\left(\mathbf{X}_j - \overline{\mathbf{X}}\right)$, $j = 1,\cdots,n$. Moreover, a parameter $\beta = \frac{1}{\sqrt{2}}\left(\frac{(2p+1)n}{4}\right)^{\frac{1}{p+4}}$ is used according to the Henze and Zirkler's suggestion.

## 3. Sample significance level and power of compared tests

In this section, the tests described in (2.1)-(2.4) and the Henze-Zirkler's test, denoted by H-Z (Henze and Zirkler, 1990) are compared via simulation study using Volfram Mathematica 9.0. The simulation studies were performed for $p = 2,3,4,5$ and $n = 8, 9, 10\,(5)\,50\,(10)\,100$ for nominal significance level $\alpha = 0.05$. For each $p$ and $n$ considered, ten thousand random samples were generated from known distribution. Sample significance level and power were calculated as proportion of all rejection of multivariate normality to 10,000.

### 3.1. Sample significance level

As all tests considered in the paper are invariant so to calculate sample significance levels, 10,000 random samples were generated from standard multivariate normal distribution. The results are presented in Figure 1.

Figure 1 shows that the Henze-Zirkler's test keeps the nominal significance level the best. For $n \geq 20$ and for all $p$ considered tests $T_M$ and $T_S$ have a little lower sample significance level than 0.05, while the Jarque-Bera's tests with Mardia's and Srivastava's sample skewness and kurtosis exceed the nominal significance level.

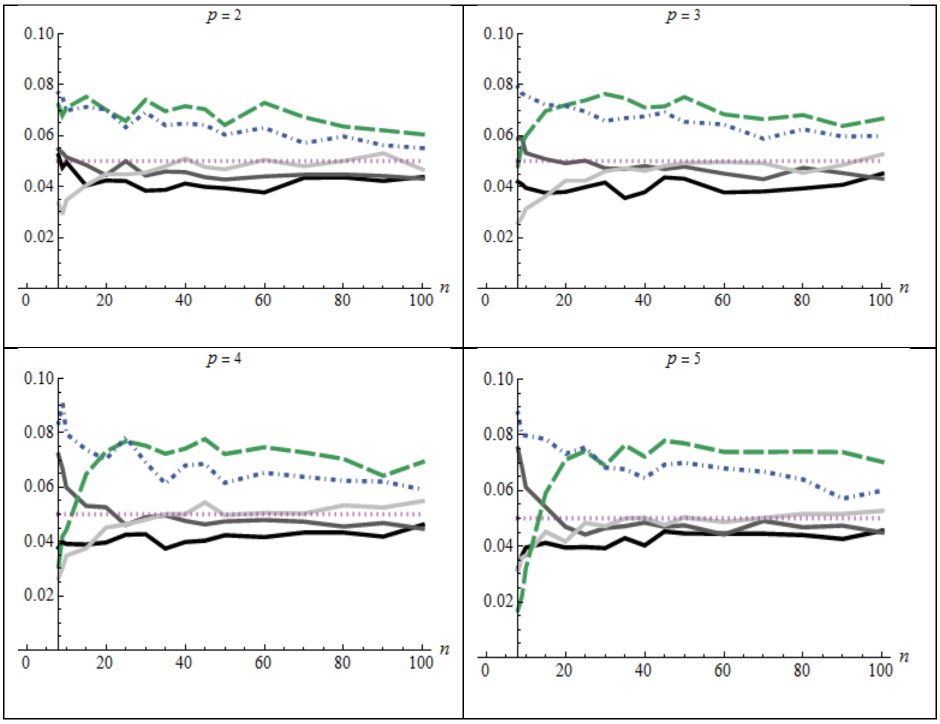


**Fig. 1**. Sample significance level of $T_M$ (——), $T_S$ (——), H-Z (——), $JB'_M$ (– –) and $JB'_S$ (– ·)

### 3.2. Power of the tests considered under chosen alternative distributions

To calculate power of tests described in (2.1)-(2.4) we consider the only one representing alternative distribution characterized by heavy tails, light tails, unsymmetrical and two-modals. From heavy tails distribution the multivariate T distribution (Pearson Type VII) with 2 degrees of freedom and the covariance $\mathbf{\Sigma} = \mathbf{I}_p + \mathbf{1}_p \mathbf{1}'_p$, where $\mathbf{I}_p$ denotes the identity matrix of the size *p*, $\mathbf{1}_p$ - vector of *p* units was taken into account. The results are enclosed in Figure 2.

Figure 2 illustrates that the tests $T_M$ and $T_S$ are less powerful in comparison with the Jarque-Bera's and the Henze-Zirkler's tests for $p = 2$. For bigger *p* the $T_M$ test has a higher power especially for $n \geq 20$. In all cases, the $T_S$ test (based on the Srivastava's skewness and kurtosis) is the weakest.

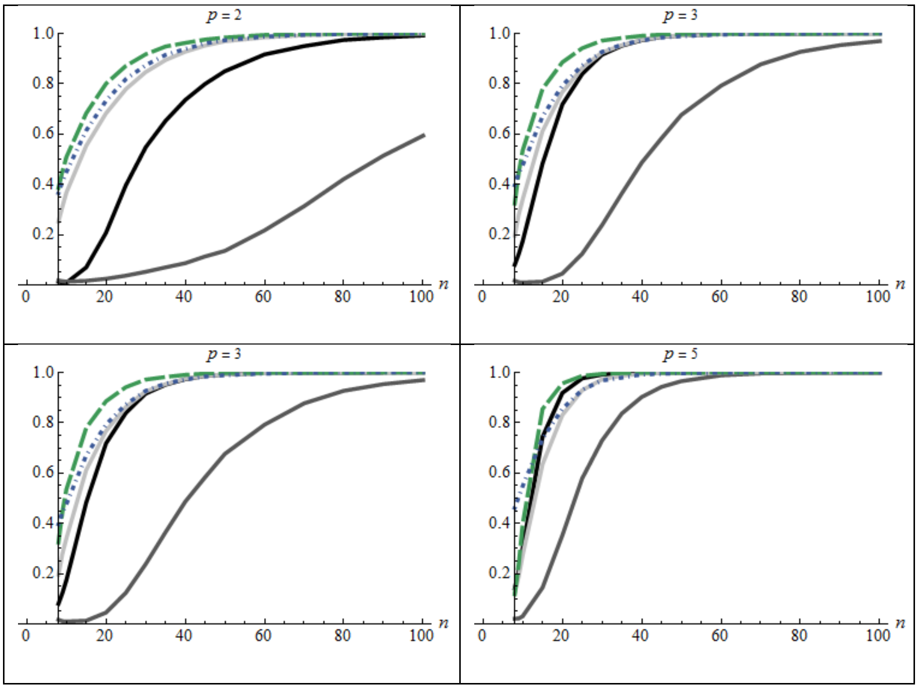


**Fig. 2.** Sample power of TM (▬▬), TS (▬▬), H-Z (▬▬), $JB'_M$ (▬ ▬) and $JB'_S$ (▬ ·) against multivariate T distribution

From the light-tailed distribution the multivariate uniform distribution on the ellipsoid (Pearson Type II with shape parameter $m = 0$) was considered. The simulation results are enclosed in Figure 3.

The last alternative distribution taken into account in the paper, is the contaminated multivariate normal distributions (a two-modal distribution), represented by $(1-\pi)N_p(\mathbf{0}_p, \mathbf{I}_p) + \pi N_p(2\mathbf{1}_p, \mathbf{I}_p + \mathbf{1}_p\mathbf{1}'_p)$, where $\pi \in [0,1]$ denotes probability of drawing from the distribution $N_p(2\mathbf{1}_p, \mathbf{I}_p + \mathbf{1}_p\mathbf{1}'_p)$ and with probability $(1-\pi)$ from $N_p(\mathbf{0}_p, \mathbf{I}_p)$, $\mathbf{0}_p$ is vector of $p$ nulls. The simulation results are shown in Figure 4.

In Figure 4 we can observe that the tests $T_M$ and $T_S$ have very small power comparing to the rest ones. The Jarque-Bera's tests and the Henze-Zirkler one are powerful in the case of when about 20 % comes from the second multivariate normal distribution.

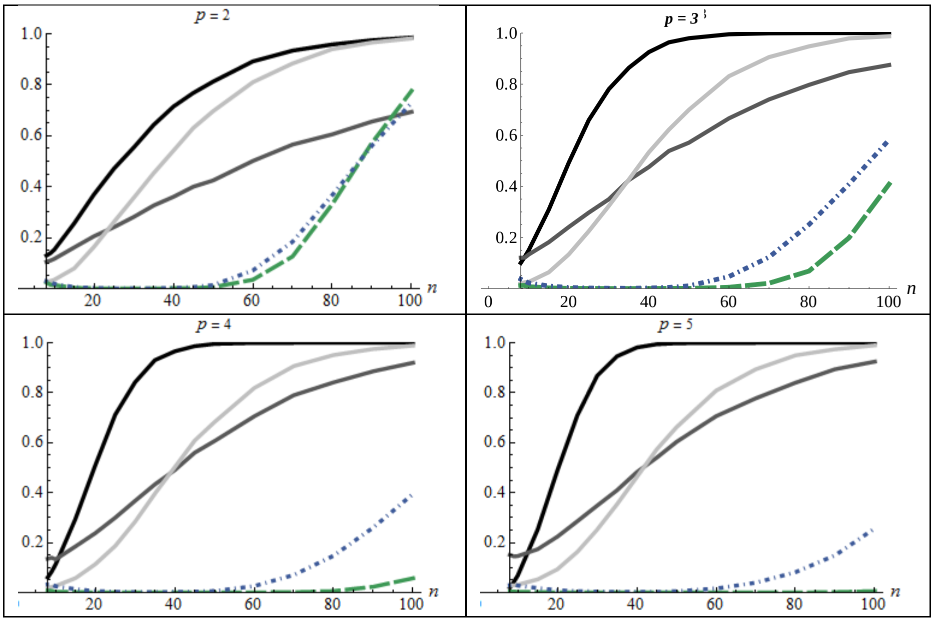


**Fig. 3.** Sample power of $T_M$ (—), $T_S$ (—), H-Z (—), $JB'_M$ (– –) and $JB'_S$ (– ·) against multivariate uniform distribution on the ellipsoid

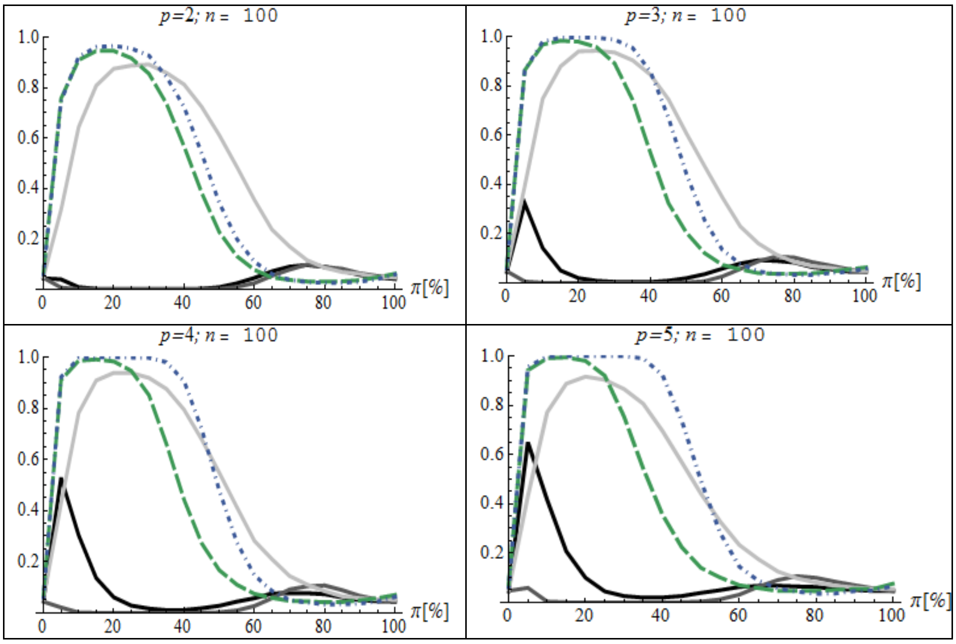


**Fig. 4.** Sample power of $T_M$ (—), $T_S$ (—), H-Z (—), $JB'_M$ (– –), $JB'_S$ (– ·) against mixture of two multivariate normal distributions

## 4. Conclusions

Proposed in the paper two tests $T_M$ and $T_S$ have an asymptotic Student's $t$-distribution with $\frac{1}{6}p(p+1)(p+2)$ and $p$ degrees of freedom, respectively. Sample significance levels of these tests are slightly below the nominal level. Power of the tests depends on the alternative distributions. $T_M$ test is the most powerful for multivariate uniform distribution on the ellipsoid. In general, for all cases considered in the paper the test $T_M$ proved to be better than $T_S$. Both tests do not recognize the mixture of two multivariate normal variates with different means and covariance matrices.

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