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CONSTRUCTION OF REGULAR D−OPTIMAL WEIGHING DESIGNS WITH NON-NEGATIVE CORRELATED ERRORS

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Summary

In this paper the problem of indicating chemical balance weighing designs providing that they satisfy the criterion of D-optimality is considered. Moreover, we study such designs under assumption that the measurements are equal positive correlated or uncorrelated and they have the same variances. We give new method of construction of D-optimal designs. They are based on the set of the incidence matrices of the balanced bipartite weighing designs. Theoretical research is illustrated by examples.

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1. Introduction

Let us consider an experiment where unknown measurements of p objects are determined in n weighing operations. In practice, however, it may not be possible to take all objects in each measurement operation. Let us assume that at most m $(\leq n)$ objects can be included in each measurement. Assume further that the results of measurements are described by the model $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$, where $\mathbf{y}$ is an $n \times 1$ random vector of the observations, $\mathbf{w} = (w_1, w_2, ..., w_p)'$ is a vector representing unknown measurements of objects, $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}\{-1, 0, 1\}$, where $\mathbf{\Phi}_{n \times p,m}\{-1, 0, 1\}$ denotes the class of $n \times p$ matrices having entries $x_{ij} = -1, 1$ or 0 when object is measured on the left pan, right pan or is not included in weighting operation, respectively. Therefore, m is the maximal number of elements equal to 1 and -1 in each column of the matrix $\mathbf{X}$. Throughout the paper we assume that $\mathbf{e}$ is an $n \times 1$ random vector of errors and $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$, $\mathrm{Var}(\mathbf{e}) = \sigma^2 \mathbf{G}$, where

$$\mathbf{G} = g\left((1-\rho)\mathbf{I}_n + \rho \mathbf{1}_n \mathbf{1}_n'\right),\ g > 0,\ 0 \le \rho < 1. \tag{1.1}$$

Note that for $0 \le \rho < 1$ and $g > 0$ the matrix $\mathbf{G}$ is positive definite. If the matrix $\mathbf{X}$ is of full column rank, then all w_j, $j = 1,2,..., p$, are estimable and the variance matrix of the best linear unbiased estimator $\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ is equal to $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$. The matrix $\mathbf{M} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix of the design $\mathbf{X}$.

Some problems related to the chemical balance weighing designs are presented in Banerjee (1975). The applications of such designs in optics are shown in Koukouvinos and Seberry J. (1997), whereas in experiments with microarrays in Banerjee and Mukerjee (2007).

In the literature a few optimality criteria minimizing some functions of matrix $\mathbf{M}$ are considered. One of them is D-optimality. We say that the design $\mathbf{X}_D$ is called D-optimal in the class of all possible design matrices $\mathbf{\Phi}_{n \times p,m}\{-1, 0, 1\}$ if

$$\det(\mathbf{X}_D) = \min\left(\det\left(\mathbf{M}^{-1}\right) : \mathbf{X} \in \mathbf{\Phi}_{n \times p,m}\{-1, 0, 1\}\right). \tag{1.2}$$

If $\det(\mathbf{X}_D)$ attains the lowest bound then the design is called regular D-optimal. In other cases it is called D-optimal. We note, each regular D-optimal design is D-optimal and the inverse sentence may not be true. The concept of D-optimality was considered in Raghavarao (1971), Shah and Sinha (1989). For the case $\mathbf{G} = \mathbf{I}_n$, Jacroux et al. (1983) presented the problems related to the D-optimality of weighing designs.

Ceranka and Graczyk (2014) formulated the definition of the regular D-optimal chemical balance weighing design and they proved the theorem presenting the conditions determining regular D-optimal designs for the case $0 \le \rho < 1$. The definition and theorem are as follows.

Definition 1.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, $0 \le \rho < 1$, is regular D-optimal if $\det\left(\mathbf{M}^{-1}\right) = \left(\frac{g(1-\rho)}{m}\right)^p$.

Theorem 1.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is regular D-optimal if and only if

(i) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ if $\rho = 0$,

(ii) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ and $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$ if $0 < \rho < 1$.

2. The construction

Some constructions of regular D-optimal designs were given in Masaro and Wong (2008) and Katulska and Smaga (2013) for the case of $\mathbf{X} \in \mathbf{\Pi}_{n\times p}\{-1, 1\}$, where $\mathbf{\Pi}_{n\times p}\{-1, 1\}$ denotes the class of $n \times p$ matrices having entries $x_{ij} = -1$ or 1. Ceranka and Graczyk (2014) considered the problem of optimality in weighing designs for the case $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$. In this Section we give new construction based on the set of the incidence matrices of balanced bipartite weighing designs. Some elementary properties of these designs are given in Huang (1976). The advantage of using balanced bipartite weighing designs lies in the fact that based on their incidence matrices we are able to construct the matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ satisfying optimality criterion.

A balanced bipartite weighing design there is an arrangement of v treatments into b blocks in such a way that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, where $k = k_1 + k_2$. Each treatment appears in r blocks, moreover each pair of treatments from different subblocks appears together in λ_1 blocks and each pair of treatments from the same subblock appears together in λ_2 blocks. The integers v, b, r, k_1, k_2, λ_1, λ_2 are called the parameters of the balanced bipartite weighing design and satisfy the following equalities

$$vr = bk, \quad b = \frac{\lambda_1 v(v-1)}{2k_1k_2}, \quad \lambda_2 = \frac{\lambda_1[k_1(k_1-1)+k_2(k_2-1)]}{2k_1k_2}, \quad r = \frac{\lambda_1 k(v-1)}{2k_1k_2}.$$

Let $\mathbf{N}_h^*$ be the incidence matrix of balanced bipartite weighing design with the parameters v, b_h, r_h, k_{1h}, k_{2h}, λ_{1h}, λ_{2h}, $h = 1,2,...,t$. From $\mathbf{N}_h^*$ we obtain another matrix $\mathbf{N}_h$ by replacing k_{1h} unities equal to +1 of each column which correspond to the elements belonging to the first subblock by -1. Accordingly, each column of the matrix $\mathbf{N}_h$ will contain k_{1h} elements equal to -1, k_{2h} elements equal to 1 and $v - k_{1h} - k_{2h}$ elements equal to 0. Let $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ be the design matrix of the chemical balance weighing design of the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \cdots & \mathbf{N}_t \end{bmatrix}'. \tag{2.1}$$

Clearly, such form of the design implies that in each weighing, from $p = v$ objects, exactly $k = k_{1h} + k_{2h}$ $(h = 1,2,...,t)$ objects are taken to the measurements, where k_{1h} of them with factor -1 and k_{2h} with factor +1. Also, each object is weighted $m = \sum_{h=1}^{t} r_h$ times in the $n = \sum_{h=1}^{t} b_h$ weighing operations.

Lemma 2.1. The chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) is nonsingular if and only if

$$k_{1h} \neq k_{2h} \tag{2.2}$$

for at least one $h = 1,2,...,t$.

Proof. If the matrix $\mathbf{G}$ is positive definite then the matrix $\mathbf{M}$ is nonsingular if and only if $\mathbf{X}'\mathbf{X}$ is nonsingular. Under the assumption that $\mathbf{G}$ is of the form (1.1), for the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given in the form (2.1), we have

$$\mathbf{X}'\mathbf{X} = \left[\sum_{h=1}^{t}(r_h - \lambda_{2h} + \lambda_{1h})\right]\mathbf{I}_v + \left[\sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h})\right]\mathbf{1}_v\mathbf{1}_v' . \tag{2.3}$$

In this way we obtain that

$$\det(\mathbf{X}'\mathbf{X}) = \left[\sum_{h=1}^{t}(r_h - \lambda_{2h} + \lambda_{1h})\right]^{v-1}\left[\sum_{h=1}^{t}(r_h + (v-1)(\lambda_{2h} - \lambda_{1h}))\right]. \tag{2.4}$$

The determinant (2.4) equals 0 if and only if

$$\sum_{h=1}^{t} r_h = \sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h}) \tag{2.5}$$

or

$$(1-v)\sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h}) = \sum_{h=1}^{t} r_h . \tag{2.6}$$

Using the relations between parameters of the balanced bipartite weighing designs (see Huang, 1976) it can be shown that (2.5) implies $v\sum_{h=1}^{t}(k_{1h} + k_{2h}) = \sum_{h=1}^{t}(k_{1h} - k_{2h})^2$, which is not satisfied, because $v \geq k_{1h} + k_{2h}$, $h = 1,2,...,t$. Under the relations between parameters of the balanced bipartite weighing designs, we can see that (2.6) implies $(k_{1h} - k_{2h})^2 = 0$ for each $h = 1,2,...,t$. The last expression does not hold if and only if $k_{1h} \neq k_{2h}$ for at least one $h = 1,2,...,t$, that finishes the proof. ■

From Theorem 1.1 we can see that the optimality conditions depend on the parameter ρ in (1.1). This implies that the methods of construction of the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ are depended on ρ, either. Therefore, next theorem ensures the existence of optimal designs.

Theorem 2.1. Let $\rho = 0$. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$, is regular D-optimal if and only if

$$\sum_{h=1}^{t}\left(\lambda_{2h} - \lambda_{1h}\right) = 0 . \qquad (2.7)$$

Proof. For the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in the form (2.1), the condition (2.3) is satisfied. From this it follows that condition (2.7) is fulfilled which is the desired conclusion. ■

We can certainly assume that $\lambda_{2h} - \lambda_{1h} = 0$ for each $h = 1,2,...,t$. Therefore under assumptions of Theorem 2.2 we obtain the following corollary.

Corollary 2.1. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$ is regular D-optimal if and only if $\lambda_{2h} = \lambda_{1h}$ for each $h = 1,2,...,t$.

We recall the following theorem given by Ceranka and Graczyk (2005) that will need in further considerations.

Theorem 2.2. If in the balanced bipartite weighing design $k_1 = 0.5s(s-1)$ and $k_2 = 0.5s(s+1)$, $s = 2,3,...$, then $\lambda_1 = \lambda_2$.

Summarizing, we can use the series of balanced bipartite weighing designs given by Huang (1976) and Ceranka and Graczyk (2005) and we can formulate the following theorem.

Theorem 2.3. The existence of the balanced bipartite weighing design with the parameters v, $b_h = \dfrac{2uv(v-1)}{s^2(s^2-1)}$, $r_h = \dfrac{2u(v-1)}{s^2-1}$, $k_{1h} = \dfrac{s(s-1)}{2}$, $k_{2h} = \dfrac{s(s+1)}{2}$, $\lambda_{1h} = \lambda_{2h} = u$, $s = 2,3,...$, $u = 1,2,...$, implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$.

Proof. It is easy to verify that the parameters of balanced bipartite weighing designs satisfy the condition (2.7). ■

It is worth pointing out that there is a big number of receivable combinations between parameters of the balanced bipartite weighing designs for that the condition $\lambda_{1h} = \lambda_{2h}$, $h = 1,2,...,t$, holds. For that reason we have many possible constructions of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ for given $p = v$ and $n = \sum_{h=1}^{t} b_h$.

For $t = 1$, the method of construction of $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ was given by Ceranka and Katulska (1999).

Our next claim is the case $t = 2$. In this case, the list of parameters of the balanced bipartite weighing design under the restrictions $v \le 25$, $b_h \le 50$, $k_{1h} + k_{2h} = 4, 9, 16$, when $\lambda_{1h} \ne \lambda_{2h}$, $h = 1, 2$, is given in Ceranka and Graczyk (2002). The condition (2.7) implies that

$$\lambda_{22} - \lambda_{12} = \lambda_{11} - \lambda_{21}. \tag{2.8}$$

We check at once that if the parameters of two balanced bipartite weighing designs satisfy the condition (2.8), then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) is regular D-optimal. We formulate theorem presenting the parameters of balanced bipartite weighing designs that satisfy the condition (2.8).

2.1. The case $\rho = 0$

Theorem 2.4. Let $\rho = 0$. The existence of the balanced bipartite weighing design with the parameters

(i) $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 3s$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $v = 2s+1$, $b_2 = s(2s+1)$, $r_2 = 7s$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$, $s = 3,4,...$,

(ii) $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 6s$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = 2s+1$, $b_2 = s(2s+1)$, $r_2 = 7s$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$, $s = 3,4,...$,

(iii) $v = 4s + 1$, $b_1 = s(4s + 1)$, $r_1 = 4s$, $k_{11} = 2$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 5s$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$, $s = 1,2,\ldots$,

(iv) $v = 4s + 1$, $b_1 = s(4s + 1)$, $r_1 = 4s$, $k_{11} = 2$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $v = 4s + 1$, $b_2 = 2s(4s + 1)$, $r_2 = 14s$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$, $s = 2,3,\ldots$,

(v) $v = 4s + 1$, $b_1 = s(4s + 1)$, $r_1 = 5s$, $k_{11} = 2$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 2$ and $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 5s$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$, $s = 2,3,\ldots$,

(vi) $v = 4s + 1$, $b_1 = s(4s + 1)$, $r_1 = 5s$, $k_{11} = 2$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 2$ and $v = 4s + 1$, $b_2 = 2s(4s + 1)$, $r_2 = 14s$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$, $s = 2,3,\ldots$,

(vii) $v = 4s + 1$, $b_1 = 2s(4s + 1)$, $r_1 = 6s$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 5s$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$, $s = 1,2,\ldots$,

(viii) $v = 4s + 1$, $b_1 = 2s(4s + 1)$, $r_1 = 12s$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 5s$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$, $s = 2,3,\ldots$,

(ix) $v = 4s + 1$, $b_1 = 2s(4s + 1)$, $r_1 = 16s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 8s$, $k_{12} = 2$, $k_{22} = 6$, $\lambda_{12} = 6$, $\lambda_{22} = 8$, $s = 2,3,\ldots$,

(x) $v = 10s + 1$, $b_1 = 5s(10s + 1)$, $r_1 = 12s$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = 10s + 1$, $b_2 = s(10s + 1)$, $r_2 = 6s$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 1$, $\lambda_{22} = 2$, $s = 1,2,...$,

(xi) $v = 10s + 1$, $b_1 = 5s(10s + 1)$, $r_1 = 15s$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $v = 10s + 1$, $b_2 = s(10s + 1)$, $r_2 = 6s$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 1$, $\lambda_{22} = 2$, $s = 1,2,...$,

(xii) $v = 20s + 1$, $b_1 = 5s(20s + 1)$, $r_1 = 20s$, $k_{11} = 2$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $v = 20s + 1$, $b_2 = 2s(20s + 1)$, $r_2 = 12s$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 1$, $\lambda_{22} = 2$, $s = 1,2,...$,

(xiii) $v = 20s + 1$, $b_1 = 5s(20s + 1)$, $r_1 = 25s$, $k_{11} = 2$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 2$ and $v = 20s + 1$, $b_2 = 2s(20s + 1)$, $r_2 = 12s$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 1$, $\lambda_{22} = 2$, $s = 1,2,...$,

(xiv) $v = 6s$, $b_1 = 6s(6s - 1)$, $r_1 = 3(6s - 1)$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 4$, $\lambda_{21} = 2$ and $v = 6s$, $b_2 = 6s(6s - 1)$, $r_2 = 7(6s - 1)$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 20$, $\lambda_{22} = 22$, $s = 2,3,...$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2 g \mathbf{I}_n$.

Proof. The parameters of the balanced bipartite weighing design (i)-(xiv) satisfy the condition (2.8). ■

2.2. The case $0<\rho<1$

Theorem 2.5. Let $0<\rho<1$ and $k_{1h} \neq k_{2h}$, $h=1,2,...,t$. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1) is regular D-optimal if and only if (2.7) holds and

$$\sum_{h=1}^{t} \frac{\lambda_{1h}}{2k_{2h}} = \sum_{h=1}^{t} \frac{\lambda_{1h}}{2k_{1h}}. \tag{2.9}$$

Proof. For the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$ in the form (2.1), from Theorem 2.2 we have (2.7). Since $0<\rho<1$ then from Theorem 2.1 we have $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$, i.e. in each column the number of elements equal to -1 is equal to the number of elements equal to 1. For $k_{1h} \neq k_{2h}$, $r_{1h} = \frac{\lambda_{1h}(v-1)}{2k_{2h}}$ and $r_{2h} = \frac{\lambda_{1h}(v-1)}{2k_{1h}}$, $h=1,2,...,t$, are the numbers of elements equal to -1 and 1 in each column of $\mathbf{N}_h'$. Hence we have (2.9). ■

There is a big number of combinations between the parameters of the balanced bipartite weighing designs for that the conditions (2.7) and (2.9) hold. Thus, we have many possible constructions of the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$ of the regular D-optimal chemical balance weighing design for given number of objects $p=v$ and measurements $n=\sum_{h=1}^{t} b_h$.

Let us consider the case $t=1$. The conditions (2.7) and (2.9) imply that $\lambda_1 = \lambda_2$ and $r_1 = r_2$. From the last one condition it follows that $k_1 = k_2$. But, when $k_1 = k_2$ we are not able to calculate r_1 and r_2. For that reason we formulate the following theorem.

Theorem 2.6. If $0<\rho<1$ then regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$ of the form $\mathbf{X} = \mathbf{N}'$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), $\mathbf{N}$ is constructed from $\mathbf{N}^*$ the incidence matrix of the balanced bipartite weighing design as is described above, does not exist.

Let us consider the case $t = 2$. Here and subsequently the existence of the balanced bipartite weighing design with the parameters v_1, b_1, r_1, k_{11}, k_{21}, λ_{11}, λ_{21} implies the existence of the balanced bipartite weighing design with the parameters $v_2 = v_1$, $b_2 = b_1$, $r_2 = r_1$, $k_{12} = k_{21}$, $k_{22} = k_{11}$, $\lambda_{12} = \lambda_{11}$, $\lambda_{22} = \lambda_{21}$. Thus we have the following theorem.

Theorem 2.7. Let $0 < \rho < 1$. For any v, the existence of the balanced bipartite weighing design with the parameters v, b_1, r_1, k_{11}, k_{21}, λ_{11}, λ_{21} for which $k_{11} \neq k_{21}$ and v, $b_2 = b_1$, $r_2 = r_1$, $k_{12} = k_{21}$, $k_{22} = k_{11}$, $\lambda_{12} = \lambda_{11}$, $\lambda_{22} = \lambda_{21}$ implies the existence of regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in the form (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1).

Proof. It is easy to show that for the parameters of the balanced bipartite weighing design given in above theorem, the conditions (2.7) and (2.9) hold. ■

Based on the result given in Theorem 2.7 let us consider

$$\mathbf{X}_h = \begin{bmatrix} \mathbf{N}_h^{'} \\ -\mathbf{N}_h^{'} \end{bmatrix}. \tag{2.10}$$

Then for each $h = 1,2,...,t$, the condition $\mathbf{X}_h^{'}\mathbf{1}_{2b} = \mathbf{0}_p$ is always true.

Theorem 2.8. Let $0 < \rho < 1$. For any h, $h = 1,2,...,t$, the existence of the balanced bipartite weighing design with the parameters $v = t$, $b_h = \frac{2ut(t-1)}{s^2(s^2-1)}$, $r_h = \frac{2u(t-1)}{s^2-1}$, $k_{1h} = \frac{s(s-1)}{2}$, $k_{2h} = \frac{s(s+1)}{2}$, $\lambda_{1h} = \lambda_{2h} = u$, $u = 1,2,...$, $s = 2,3,...$, $t = 3,4,...$, implies the existence of regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in the form (2.1) for $\mathbf{X}_h$ in (2.10) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1).

Proof. It is easy to verify that for $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$, if the parameters of the balanced bipartite weighing designs are given as above then they fulfill the conditions (2.7) and (2.9). ■

3. Examples

To illustrate the theory given in the previous section let us consider two experiments where unknown measurements of $p = v = 5$ objects are determined in $n = 20$ measurement operations. Among all possible variance matrices $\sigma^2\mathbf{G}$ we take two $\mathbf{G} = g\mathbf{I}_n$ for $\rho = 0$ and $\mathbf{G} = g\left((1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n^{'}\right)$ for $g > 0$, $0 < \rho < 1$.

3.1. The case $\rho = 0$

Let $\rho = 0$. We assume that each object can be measured $m = 17$ times. We choose the design matrix $\mathbf{X}$ among many at our disposal in the class $\mathbf{\Phi}_{20\times5,17}\{-1, 0, 1\}$ in such a manner that $\mathbf{X}$ is regular D-optimal. Under conditions stated above the design $\mathbf{X} \in \mathbf{\Phi}_{20\times5,17}\{-1, 0, 1\}$ in the form (2.1) could be constructed from three incidence matrices of balanced bipartite weighing designs with the parameters $v = 5$, $b_1 = 10$, $r_1 = 8$, $k_{11} = 1$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 3$ and

$$\mathbf{N}_1^* = \begin{bmatrix} 0 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 \\ 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 1_1 & 1_2 \\ 1_2 & 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 1_1 \\ 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 & 0 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 & 0 \end{bmatrix},$$

where 1_1 and 1_2 denote that the object exists in the first or in the second subblock, respectively, 0 the object does not exist in the block, $v = 5$, $b_2 = 5$, $r_2 = 4$, $k_{12} = 2$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$ and $v = 5$, $b_3 = 5$, $r_3 = 5$, $k_{13} = 1$, $k_{23} = 4$, $\lambda_{13} = 2$, $\lambda_{23} = 3$, where

$$\mathbf{N}_2^* = \begin{bmatrix} 1_1 & 1_1 & 1_2 & 1_1 & 0 \\ 1_1 & 1_2 & 1_1 & 0 & 1_2 \\ 1_2 & 1_1 & 0 & 1_2 & 1_2 \\ 1_2 & 0 & 1_1 & 1_1 & 1_1 \\ 0 & 1_2 & 1_2 & 1_2 & 1_1 \end{bmatrix}, \quad \mathbf{N}_3^* = \begin{bmatrix} 1_1 & 1_2 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_1 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_2 & 1_1 & 1_1 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_2 & 1_1 \end{bmatrix}.$$

In each incidence matrix of balanced bipartite weighing design we replace the elements that are equal to 1 and correspond to elements belonging to the first subblock (1_1) by -1. As the next step we built design $\mathbf{X} \in \mathbf{\Phi}_{20\times 5,17}\{-1,0,1\}$ in the form (2.1) for $t=3$

$$\mathbf{X}' = \begin{bmatrix} 0 & 1 & 1 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 0 & -1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 0 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & 0 & 1 & 1 & 0 & -1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

and for this design $\mathbf{X}'\mathbf{X} = 17\mathbf{I}_5$ and $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1} = \left(\frac{g}{17}\right)^5$.

3.2. The case $0 < \rho < 1$

Let $0<\rho<1$. Let assume that each object can be measured $m=12$ times. We choose the design matrix $\mathbf{X}$ among many at our disposal in the class $\mathbf{\Phi}_{20\times 5,12}\{-1,0,1\}$ in such a manner that $\mathbf{X}$ is regular D-optimal. The design $\mathbf{X} \in \mathbf{\Phi}_{20\times 5,12}\{-1,0,1\}$ in the form (2.1) could be constructed from two incidence matrices of balanced bipartite weighing designs with the parameters $v=5$, $b_1=10$, $r_1=6$, $k_{11}=1$, $k_{21}=2$, $\lambda_{11}=2$, $\lambda_{21}=1$ and $v=5$, $b_2=10$, $r_2=6$, $k_{12}=2$, $k_{22}=1$, $\lambda_{12}=2$, $\lambda_{22}=1$

$$\mathbf{N}_1^* = \begin{bmatrix} 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 0 & 0 & 1_1 & 0 \\ 1_2 & 0 & 0 & 1_1 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 0 \\ 1_1 & 1_2 & 0 & 0 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_1 \\ 0 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_1 & 1_2 & 0 & 1_2 \\ 0 & 0 & 1_1 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 1_2 \end{bmatrix},$$

$$\mathbf{N}_2^* = \begin{bmatrix} 1_1 & 1_1 & 1_1 & 1_1 & 0 & 1_2 & 0 & 0 & 1_2 & 0 \\ 1_1 & 0 & 0 & 1_2 & 1_1 & 1_1 & 1_1 & 1_2 & 0 & 0 \\ 1_2 & 1_1 & 0 & 0 & 1_1 & 0 & 0 & 1_1 & 1_1 & 1_2 \\ 0 & 1_2 & 1_1 & 0 & 0 & 1_1 & 1_2 & 1_1 & 0 & 1_1 \\ 0 & 0 & 1_2 & 1_1 & 1_2 & 0 & 1_1 & 0 & 1_1 & 1_1 \end{bmatrix}$$

As the next step we built design $\mathbf{X} \in \mathbf{\Phi}_{20\times 5,17}\{-1,0,1\}$ in the form (2.1) for $t=2$ and for this design $\mathbf{X}'\mathbf{X} = 12\mathbf{I}_5$ and $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1} = \left(\frac{g(1-\rho)}{12}\right)^5$, where

$$\mathbf{X}' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & -1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}.$$

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