Colloquium Biometricum 44
2014, 5−23

ESTIMATORS OF UNKNOWN PARAMETERS IN THE GROWTH CURVE MODEL WITH THE UNIFORM CORRELATION STRUCTURE BASED ON DIFFERENT ESTIMATORS OF VARIANCE MATRIX

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Summary

We derive and compare different estimators of unknown parameters in the growth curve model with a uniform correlation structure based on various estimators of variance matrix. We deal with uniformly minimum variance unbiased invariant estimator (UMVUIE), maximum likelihood estimator and outer product estimator of variance matrix. In these cases we also show the orthogonal decomposition.

Keywords and phrases: growth curve model, uniform correlation structure, estimators of variance parameters, maximum likelihood estimator, outer product estimator, orthogonal decomposition

Classification AMS: 15A63, 62H12, 62H20

1. Introduction

The growth curve model is a generalized multivariate analysis of variance model, which was introduced by Potthoff and Roy (1964). Standard model is of the form

$$\mathbf{Y} = \mathbf{XBZ} + \boldsymbol{\varepsilon}, \quad \mathrm{vec}(\boldsymbol{\varepsilon}) : N(0, \boldsymbol{\Sigma} \otimes \mathbf{I}),$$

where $\mathbf{Y}_{n\times p}$ is matrix of observations, $\mathbf{X}_{n\times m}$ is ANOVA matrix (so that $\mathbf{1} \in \mathrm{P}(\mathbf{X})$), $\mathbf{B}_{m\times r}$ is matrix of unknown parameters, $\mathbf{Z}_{r\times p}$ is matrix of regression constants and in all article we will consider that $\mathbf{1} \in \mathrm{P}(\mathbf{Z}')$, $\boldsymbol{\varepsilon}_{n\times p}$ is matrix of random errors which has the normal distribution and $\boldsymbol{\Sigma}_{p\times p}$ is variance matrix of rows of matrix $\mathbf{Y}$. The vec operator stacks elements of a matrix into a vector column-wise and $\mathbf{I}_{n\times n}$ denotes identity matrix. In the most applications of the model, p is the number of time points observed on each of the n subjects and m is the number of groups. This model is useful especially for modeling growths of living organisms and it represents a statistical model widely used in many fields of study such as biology, medicine, economy. There are many special cases of this model depending on the correlation structure. We will deal with uniform correlation structure, which has the following form

$$\boldsymbol{\Sigma} = \sigma^2\left[(1-\rho)\mathbf{I} + \rho\mathbf{11'}\right], \tag{1.1}$$

where $\sigma^2 > 0$ and $-\frac{1}{p-1} < \rho < 1$ are unknown parameters, which we are interested in.

Let us denote $\mathbf{P}_{\mathrm{X}}$ orthogonal projector on column space $\mathrm{P}(\mathbf{X})$ of a matrix $\mathbf{X}$ and $\mathbf{M}_{\mathrm{X}} = \mathbf{I} - \mathbf{P}_{\mathrm{X}}$ orthogonal projector on its orthogonal complement. The important result is orthogonal decomposition in the growth curve model. Chinese mathematicians Jianhua Hu, Ren-Dao Ye and Song-Gui Wang came up with this idea in 2009. A simple transformation to change a model into an equivalent which allows to determine explicit forms of estimators. Let us consider generalized uniform correlation structure $\boldsymbol{\Sigma} = \theta_1\mathbf{G} + \theta_2\mathbf{ww}'$. Ye and Wang (2009) examine the model $\mathbf{YG}^{-\frac{1}{2}} = \mathbf{YG}^{-\frac{1}{2}}\mathbf{P}_{\mathbf{F}} + \mathbf{YG}^{-\frac{1}{2}}\mathbf{M}_{\mathbf{F}}$, where $\mathbf{F} = \mathbf{G}^{-\frac{1}{2}}\mathbf{w}$. Many tasks, which are very difficult or impossible to handle in basic models, can be done with ease in models consisting of mutually orthogonal components. Note that generalized uniform correlation structure with $\mathbf{G} = \mathbf{I}$, $\mathbf{w} = \mathbf{1}$, $\theta_1 = \sigma^2(1-\rho)$ and $\theta_2 = \sigma^2\rho$ is reduced to structure (1.1).

2. Estimators based on UMVUIE of $\boldsymbol{\Sigma}$

Under normality, uniformly minimum variance unbiased invariant estimator of $\boldsymbol{\Sigma}$ is defined as

$$\mathbf{S} = \frac{1}{n - r(\mathbf{X})} \mathbf{Y}'\mathbf{M}_{\mathbf{X}}\mathbf{Y}.$$

By moment method, since $\mathrm{E}(\mathbf{S}) = \boldsymbol{\Sigma}$, unbiased estimating equations are

$$\mathrm{Tr}(\mathbf{S}) - p\hat{\sigma}^2 = 0 \quad \text{and} \quad \mathbf{1}'\mathbf{S}\,\mathbf{1} - \mathrm{Tr}(\mathbf{S})\left[1 + (p-1)\hat{\rho}\right] = 0,$$

so estimators of unknown parameters are of the form

$$\hat{\sigma}^2{}_{\mathrm{Z}} = \frac{\mathrm{Tr}(\mathbf{S})}{p} \quad \text{and} \quad \hat{\rho}_{\mathrm{Z}} = \frac{1}{p-1}\left(\frac{\mathbf{1}'\mathbf{S}\,\mathbf{1}}{\mathrm{Tr}(\mathbf{S})} - 1\right). \tag{2.1}$$

These estimators were derived by Žežula (2006). Now we consider the modified model with orthogonal decomposition proposed by Ye and Wang

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 = \mathbf{Y}\mathbf{P}_{\mathbf{1}} + \mathbf{Y}\mathbf{M}_{\mathbf{1}},$$

where $\mathbf{1} = (1, \ldots, 1)'$, $\mathbf{P}_{\mathbf{1}} = \frac{1}{p}\mathbf{1}\mathbf{1}'$, $\mathbf{M}_{\mathbf{1}} = \mathbf{I} - \mathbf{P}_{\mathbf{1}}$.

If we denote

$$\mathbf{V}_1 = \mathbf{P}_1\mathbf{S}\mathbf{P}_1 = \sigma^2\left((1-\rho)\frac{1}{p} + \rho\right)\mathbf{1}\mathbf{1}',$$

$$\mathbf{V}_2 = \mathbf{M}_1\mathbf{S}\mathbf{M}_1 = \sigma^2(1-\rho)\left(\mathbf{I} - \frac{1}{p}\mathbf{1}\mathbf{1}'\right),$$

then estimators of unknown parameters are of the form

$$\hat{\sigma}^2{}_{\mathrm{YW}} = \frac{\mathrm{Tr}(\mathbf{V}_1) + \mathrm{Tr}(\mathbf{V}_2)}{p}, \quad \hat{\rho}_{\mathrm{YW}} = 1 - \frac{p}{p-1} \cdot \frac{\mathrm{Tr}(\mathbf{V}_2)}{\mathrm{Tr}(\mathbf{V}_1) + \mathrm{Tr}(\mathbf{V}_2)}. \tag{2.2}$$

Klein and Žežula (2010) showed that $\hat{\sigma}^2{}_{\mathrm{Z}} = \hat{\sigma}^2{}_{\mathrm{YW}}$ and $\hat{\rho}_{\mathrm{Z}} = \hat{\rho}_{\mathrm{YW}}$.

Now we will have a look at estimators of unknown parameters for maximum likelihood estimator of $\boldsymbol{\Sigma}$ and for outer product estimator of the variance matrix.

3. Estimators based on maximum likelihood estimator of $\boldsymbol{\Sigma}$

Maximum likelihood estimator of matrix $\boldsymbol{\Sigma}$ is in the form

$$\mathbf{S^M} = \frac{1}{n}\left(\mathbf{Y}'\mathbf{M_X}\mathbf{Y} + \mathbf{M_{Z'}}\mathbf{Y}'\mathbf{P_X}\mathbf{Y}\mathbf{M_{Z'}}\right).$$

The MLE of σ^2 and ρ are defined as above, but $\mathbf{S}$ is replaced by $\mathbf{S^M}$. Thus

$$\hat{\sigma}^2{}_{\mathrm{M}} = \frac{\mathrm{Tr}(\mathbf{S^M})}{p} \quad \text{and} \quad \hat{\rho}_{\mathrm{M}} = \frac{1}{p-1}\left(\frac{\mathbf{1}'\mathbf{S^M}\mathbf{1}}{\mathrm{Tr}(\mathbf{S^M})} - 1\right). \tag{3.1}$$

For derivation of the estimators based on unbiased estimating equations in this case we will need $\mathrm{E}(\mathrm{Tr}(\mathbf{S^M}))$ and $\mathrm{E}(\mathbf{1}'\mathbf{S^M}\mathbf{1})$. The first matrix in $\mathbf{S^M}$ is the matrix $\mathbf{S}$ except for a constant. The problem is the second matrix. Knowing that

$$\mathbf{Y} : N_{n\times p}(\mathbf{XBZ}, \boldsymbol{\Sigma}, \mathbf{I}) \quad \Rightarrow \quad \mathrm{vec}\,\mathbf{Y} : N_{np}\left(\mathrm{vec}(\mathbf{XBZ}), \boldsymbol{\Sigma} \otimes \mathbf{I}\right) \tag{3.2}$$

we can write

$$\mathrm{Var}\left[(\mathbf{M_{Z'}} \otimes \mathbf{I})\mathrm{vec}(\mathbf{Y})\right] = \mathbf{M_{Z'}}\boldsymbol{\Sigma}\mathbf{M_{Z'}} \otimes \mathbf{I},$$

so

$$\mathrm{vec}\,(\mathbf{Y}\mathbf{M_{Z'}}) : N_{np}\left(\mathbf{0}, \mathbf{M_{Z'}}\boldsymbol{\Sigma}\mathbf{M_{Z'}} \otimes \mathbf{I}\right).$$

In the next step we will need the expected value theorem derived by Ghazal and Neudecker (2000). Let us consider matrix $\mathbf{K} = (\mathbf{T}_{n\times p}, \mathbf{U}_{n\times q})$, with $\mathrm{E}(\mathbf{K}) = (\mathbf{M}, \mathbf{N})$ and

$$\mathrm{Var}(\mathrm{vec}\,\mathbf{K}) = \boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\Omega}_{TT} & \boldsymbol{\Omega}_{TU} \\ \boldsymbol{\Omega}_{UT} & \boldsymbol{\Omega}_{UU} \end{bmatrix}.$$

Let further $\boldsymbol{\Omega} = \boldsymbol{\Phi} \otimes \mathbf{I}_n$, with

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Phi}_{TT} & \boldsymbol{\Phi}_{TU} \\ \boldsymbol{\Phi}_{UT} & \boldsymbol{\Phi}_{UU} \end{bmatrix}.$$

Ghazal and Neudecker showed that

$$\mathrm{E}(\mathbf{T}'\mathbf{AU}) = \mathrm{Tr}(\mathbf{A})\mathbf{\Omega}_{TU} + \mathbf{M}'\mathbf{AN}.$$

In particular,

$$\mathrm{E}(\mathbf{T}'\mathbf{AT}) = \mathrm{Tr}(\mathbf{A})\mathbf{\Omega}_{TT} + \mathbf{M}'\mathbf{AM}. \tag{3.3}$$

On the basis of (3.3) and the well known fact that $\mathrm{Tr}(\mathbf{P_X}) = r(\mathbf{X})$, $\mathrm{Tr}(\mathbf{M_X}) = n - r(\mathbf{X})$ and $\mathbf{M_X 1} = \mathbf{0}$ holds following

$$\begin{aligned}\mathrm{E}(\mathbf{S^M}) &= \frac{n-r(\mathbf{X})}{n}\mathrm{E}(\mathbf{S}) + \frac{1}{n}\mathrm{E}\left(\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{P_X}\mathbf{Y}\mathbf{M}_{\mathbf{Z}'}\right)\\ &= \frac{n-r(\mathbf{X})}{n}\mathbf{\Sigma} + \frac{1}{n}r(\mathbf{X})\mathbf{M}_{\mathbf{Z}'}\mathbf{\Sigma}\mathbf{M}_{\mathbf{Z}'}.\end{aligned}$$

Consequently

$$\begin{aligned}\mathrm{E}\left(\mathrm{Tr}\left(\mathbf{S^M}\right)\right) &= \frac{n-r(\mathbf{X})}{n}p\sigma^2 + \frac{1}{n}r(\mathbf{X})\mathrm{Tr}\left(\mathbf{M}_{\mathbf{Z}'}\sigma^2\left((1-\rho)\mathbf{I} + \rho\mathbf{11}'\right)\right)\\ &= \frac{n-r(\mathbf{X})}{n}p\sigma^2 + \frac{r(\mathbf{X})}{n}\sigma^2(1-\rho)(p - r(\mathbf{Z})).\end{aligned}$$

Similarly as above we can write

$$\mathrm{E}\left(\mathbf{1}'\mathbf{S^M}\mathbf{1}\right) = \frac{n-r(\mathbf{X})}{n}\mathrm{E}\left(\mathbf{1}'\mathbf{S}\mathbf{1}\right) = \frac{n-r(\mathbf{X})}{n}p\left(1+(p-1)\rho\right)\sigma^2.$$

Based on the unbiased estimating equations we can derive that these unbiased estimators of unknown parameters in this case are

$$\begin{aligned}\hat{\sigma}^2{}_{\mathrm{B}} &= \frac{p(p-1)n(n-r(\mathbf{X}))\mathrm{Tr}(\mathbf{S^M}) + nr(\mathbf{X})(p-r(\mathbf{Z}))\mathbf{1}'\mathbf{S^M}\mathbf{1}}{p^2(n-r(\mathbf{X}))\left[(n-r(\mathbf{X}))(p-1) + r(\mathbf{X})(p-r(\mathbf{Z}))\right]},\\ \hat{\rho}_{\mathrm{B}} &= \frac{1}{p-1}\left(\frac{p\left[(n-r(\mathbf{X}))(p-1) + r(\mathbf{X})(p-r(\mathbf{Z}))\right]\mathbf{1}'\mathbf{S^M}\mathbf{1}}{(n-r(\mathbf{X}))p(p-1)\mathrm{Tr}(\mathbf{S^M}) + r(\mathbf{X})(p-r(\mathbf{Z}))\mathbf{1}'\mathbf{S^M}\mathbf{1}} - 1\right).\end{aligned} \tag{3.4}$$

Now we use the orthogonal decomposition for this case. First modify matrix $\mathbf{S^M}$ to the form

$$\mathbf{S}^{\mathbf{M}} = \frac{n-r(\mathbf{X})}{n}\mathbf{S} + \frac{1}{n}\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{P}_{\mathbf{X}}\mathbf{Y}\mathbf{M}_{\mathbf{Z}'}.$$

Let us denote

$$\mathbf{V}_1^{\mathbf{M}} = \mathbf{P}_1\mathbf{S}^{\mathbf{M}}\mathbf{P}_1 \quad \text{and} \quad \mathbf{V}_2^{\mathbf{M}} = \mathbf{M}_1\mathbf{S}^{\mathbf{M}}\mathbf{M}_1.$$

For traces of these matrices applies

$$\begin{aligned}\operatorname{Tr}\left(\mathbf{V}_1^{\mathrm{M}}\right) &= \frac{n-r(\mathbf{X})}{n}\operatorname{Tr}\left(\mathbf{P}_1\mathbf{S}\mathbf{P}_1\right) + \frac{1}{n}\operatorname{Tr}\left(\mathbf{P}_1\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{P}_{\mathbf{X}}\mathbf{Y}\mathbf{M}_{\mathbf{Z}'}\mathbf{P}_1\right) \\ &= \frac{n-r(\mathbf{X})}{n}\frac{1}{p}\mathbf{1}'\mathbf{S}\mathbf{1},\end{aligned}$$

$$\operatorname{Tr}\left(\mathbf{V}_2^{\mathbf{M}}\right) = \operatorname{Tr}\left(\mathbf{M}_1\mathbf{S}^{\mathbf{M}}\right) = \operatorname{Tr}\left(\mathbf{S}^{\mathbf{M}}\right) - \operatorname{Tr}\left(\mathbf{P}_1\mathbf{S}^{\mathbf{M}}\right),$$

which implies

$$\mathrm{E}\left(\operatorname{Tr}\left(\mathbf{V}_1^{\mathbf{M}}\right)\right) = \frac{n-r(\mathbf{X})}{n}\sigma^2\left[1+(p-1)\rho\right],$$

$$\begin{aligned}\mathrm{E}\left(\operatorname{Tr}\left(\mathbf{V}_2^{\mathbf{M}}\right)\right) &= \frac{n-r(\mathbf{X})}{n}p\sigma^2 + \frac{r(\mathbf{X})}{n}\sigma^2(1-\rho)(p-r(\mathbf{Z})) - \frac{1}{p}\mathrm{E}\left(\mathbf{1}'\mathbf{S}^{\mathbf{M}}\mathbf{1}\right) \\ &= \frac{n-r(\mathbf{X})}{n}p\sigma^2 + \frac{r(\mathbf{X})}{n}\sigma^2(1-\rho)(p-r(\mathbf{Z})) - \frac{n-r(\mathbf{X})}{n}\sigma^2\left[1+(p-1)\rho\right].\end{aligned}$$

Unbiased estimating equations are

$$\operatorname{Tr}\left(\hat{\mathbf{V}}_1^{\mathbf{M}}\right) - \frac{n-r(\mathbf{X})}{n}\hat{\sigma}^2\left[1+(p-1)\hat{\rho}\right] = 0$$

and

$$\operatorname{Tr}\left(\hat{\mathbf{V}}_2^{\mathbf{M}}\right) - \frac{r(\mathbf{X})}{n}\hat{\sigma}^2(1-\hat{\rho})(p-r(\mathbf{Z})) + \frac{n-r(\mathbf{X})}{n}\hat{\sigma}^2\left[1+(p-1)\hat{\rho}-p\right] = 0.$$

If we denote

$$a = 1 + \frac{r(\mathbf{X})(p-r(\mathbf{Z}))}{(p-1)(n-r(\mathbf{X}))}$$

then estimators of unknown parameters have the form

$$\hat{\sigma}^2_B = \frac{n}{p(n-r(\mathbf{X}))}\left[\mathrm{Tr}\left(\mathbf{V}_1^{\mathbf{M}}\right) + \frac{\mathrm{Tr}\left(\mathbf{V}_2^{\mathbf{M}}\right)}{a}\right],$$
$$\hat{\rho}_B = 1 - \frac{p}{p-1}\cdot\frac{\mathrm{Tr}\left(\mathbf{V}_2^{\mathbf{M}}\right)}{a\mathrm{Tr}\left(\mathbf{V}_1^{\mathbf{M}}\right) + \mathrm{Tr}\left(\mathbf{V}_2^{\mathbf{M}}\right)}. \tag{3.5}$$

4. Estimators based on outer product estimator of $\mathbf{\Sigma}$

Another estimator of matrix $\mathbf{\Sigma}$ is defined as

$$\mathbf{S}^{\mathrm{O}} = \frac{1}{n-r(\mathbf{X})}\mathbf{Y}'\mathbf{M}_{\mathbf{X}}\mathbf{Y} + \frac{1}{n}\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{Y}\mathbf{M}_{\mathbf{Z}'} - \frac{1}{n-r(\mathbf{X})}\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{M}_{\mathbf{X}}\mathbf{Y}\mathbf{M}_{\mathbf{Z}'},$$

it is called the outer product estimator or quadratic least squares estimator of $\mathbf{\Sigma}$. Idea of derivation this estimator of variance matrix is as follows. Let us denote $\mathbf{y} = \mathrm{vec}(\mathbf{Y})$, $\mathbf{W} = \mathbf{Z}' \otimes \mathbf{X}$, $\boldsymbol{\beta} = \mathrm{vec}(\mathbf{B})$, and $\boldsymbol{\varepsilon} = \mathrm{vec}(\boldsymbol{\varepsilon})$. Then, assuming normality of observations, the growth curve model can be written as one-dimensional model

$$\mathbf{y} = \mathbf{W}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} : N_{np}(\mathbf{0}, \mathbf{\Sigma} \otimes \mathbf{I}).$$

Matrix $\mathbf{S}^{\mathrm{O}}$ is derived on the basis of the induced linear model

$$\left\{\mathbf{M}_{\mathbf{X}}\mathbf{y}\mathbf{y}'\mathbf{M}_{\mathbf{X}} = \mathrm{E}\left(\mathbf{M}_{\mathbf{X}}\mathbf{y}\mathbf{y}'\mathbf{M}_{\mathbf{X}}\right), \mathrm{e}\left(\mathbf{M}_{\mathbf{X}}\mathbf{y}\mathbf{y}'\mathbf{M}_{\mathbf{X}}\right)\right\}.$$

For its exact derivation see Wu et al. (2009). Now we show the estimators of unknown parameters for this estimator of variance matrix. We will use the already used relations. Accordingly for matrix $\mathbf{S}^{\mathrm{O}}$ holds the following

$$\begin{aligned}
\mathrm{E}\left(\mathrm{Tr}(\mathbf{S}^{\mathrm{O}})\right) &= \mathrm{E}\left(\mathrm{Tr}(\mathbf{S})\right) + \frac{1}{n}\mathrm{E}\left(\mathrm{Tr}\left(\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{Y}\mathbf{M}_{\mathbf{Z}'}\right)\right) - \frac{1}{n-r(\mathbf{X})}\mathrm{E}\left(\mathrm{Tr}\left(\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{M}_{\mathbf{X}}\mathbf{Y}\mathbf{M}_{\mathbf{Z}'}\right)\right) \\
&= p\sigma^2 + \frac{1}{n}n\sigma^2(1-\rho)(p-r(\mathbf{Z})) - \frac{1}{n-r(\mathbf{X})}(n-r(\mathbf{X}))\sigma^2(1-\rho)(p-r(\mathbf{Z})) \\
&= p\sigma^2,
\end{aligned}$$

$$\mathrm{E}\left(\mathbf{1}'\mathbf{S}^{\mathrm{O}}\mathbf{1}\right) = \mathrm{E}\left(\mathbf{1}'\mathbf{S}\mathbf{1}\right) = p\sigma^2\left(1 + (p-1)\rho\right).$$

Thus unbiased estimating equations are

$$\mathrm{Tr}(\mathbf{S}^{\mathrm{O}}) - p\hat{\sigma}^2 = 0 \quad \text{and} \quad \mathbf{1}'\mathbf{S}^{\mathrm{O}}\mathbf{1} - p\left(1 + (p-1)\hat{\rho}\right)\hat{\sigma}^2 = 0,$$

which imply estimators of the form

$$\hat{\sigma}^2{}_{\mathrm{O}} = \frac{\mathrm{Tr}(\mathbf{S}^{\mathrm{O}})}{p} \quad \text{and} \quad \hat{\rho}_{\mathrm{O}} = \frac{1}{p-1}\left(\frac{\mathbf{1}'\mathbf{S}^{\mathrm{O}}\mathbf{1}}{\mathrm{Tr}(\mathbf{S}^{\mathrm{O}})} - 1\right). \tag{4.1}$$

These estimators have the same form as Z-estimators and relations between them will be shown in the next section.

Now we use the Ye-Wang's way of estimating. Let us denote

$$\mathbf{V}_1^{\mathrm{O}} = \mathbf{P_1}\mathbf{S}^{\mathrm{O}}\mathbf{P_1} \quad \text{and} \quad \mathbf{V}_2^{\mathrm{O}} = \mathbf{M_1}\mathbf{S}^{\mathrm{O}}\mathbf{M_1}.$$

Then

$$\begin{aligned}\mathrm{Tr}(\mathbf{V}_1^{\mathrm{O}}) = {} & \frac{1}{p(n-r(\mathbf{X}))}\mathrm{Tr}\left(\mathbf{11}'\mathbf{Y}'\mathbf{M_X}\mathbf{Y}\right) + \frac{1}{np}\mathrm{Tr}\left(\mathbf{11}'\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{Y}\mathbf{M}_{\mathbf{Z}'}\right) - \\ & - \frac{1}{p(n-r(\mathbf{X}))}\mathrm{Tr}\left(\mathbf{11}'\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{M_X}\mathbf{Y}\mathbf{M}_{\mathbf{Z}'}\right) \\ = {} & \frac{1}{p(n-r(\mathbf{X}))}\mathbf{1}'\mathbf{Y}'\mathbf{M_X}\mathbf{Y1}\end{aligned}$$

consequently

$$\mathrm{E}\left(\mathrm{Tr}(\mathbf{V}_1^{\mathrm{O}})\right) = \left(1 + (p-1)\rho\right)\sigma^2.$$

Likewise

$$\mathrm{Tr}(\mathbf{V}_2^{\mathrm{O}}) = \mathrm{Tr}\left((\mathbf{I} - \mathbf{P_1})\mathbf{S}^{\mathrm{O}}\right) = \mathrm{Tr}(\mathbf{S}^{\mathrm{O}}) - \frac{1}{p}\mathbf{1}'\mathbf{S}^{\mathrm{O}}\mathbf{1},$$

which implies

$$\mathrm{E}\left(\mathrm{Tr}(\mathbf{V}_2^{\mathrm{O}})\right) = (p-1)\sigma^2(1-\rho),$$

so unbiased estimating equations are

$$\mathrm{Tr}\left(\hat{\mathbf{V}}_1^{\mathrm{O}}\right) - \left(1 + (p-1)\hat{\rho}\right)\hat{\sigma}^2 = 0 \quad \text{and} \quad \mathrm{Tr}\left(\hat{\mathbf{V}}_2^{\mathrm{O}}\right) - (p-1)\hat{\sigma}^2(1-\hat{\rho}) = 0.$$

These equations imply estimators in the form

$$\hat{\sigma}^2{}_{\mathrm{O}} = \frac{\mathrm{Tr}(\mathbf{V}_1^{\mathrm{O}}) + \mathrm{Tr}(\mathbf{V}_2^{\mathrm{O}})}{p} \quad \text{and} \quad \hat{\rho}_{\mathrm{O}} = 1 - \frac{p}{p-1}\cdot\frac{\mathrm{Tr}(\mathbf{V}_2^{\mathrm{O}})}{\mathrm{Tr}(\mathbf{V}_1^{\mathrm{O}}) + \mathrm{Tr}(\mathbf{V}_2^{\mathrm{O}})}. \tag{4.2}$$

In the next step we derive properties of matrix $\mathbf{S}^{\mathrm{O}}$ and related estimators (4.1) of unknown parameters. We need the next theorem derived by Ghazal and Neudecker (2000).

Theorem. Consider $\operatorname{vec}\mathbf{L} : N_{np}(\operatorname{vec}\mathbf{M}, \mathbf{V}\otimes\mathbf{U})$ and define $\mathbf{S}_{\mathbf{A}} = \mathbf{L}'\mathbf{A}\mathbf{L}$ and $\mathbf{S}_{\mathbf{B}}=\mathbf{L}'\mathbf{B}\mathbf{L}$. Then for covariance of $\operatorname{vec}\mathbf{S}_{\mathbf{A}}$ and $\operatorname{vec}\mathbf{S}_{\mathbf{B}}$ applies

$$\begin{aligned}\operatorname{Cov}(\operatorname{vec}\mathbf{S}_{\mathbf{A}}, \operatorname{vec}\mathbf{S}_{\mathbf{B}}) = \; & \operatorname{Tr}(\mathbf{A}'\mathbf{U}\mathbf{B}\mathbf{U})\mathbf{V}\otimes\mathbf{V} + \mathbf{M}'\mathbf{A}'\mathbf{U}\mathbf{B}\mathbf{M}\otimes\mathbf{V} + \\ & + \mathbf{V}\otimes\mathbf{M}'\mathbf{A}\mathbf{U}\mathbf{B}'\mathbf{M} + \mathbf{K}_{pp}[\operatorname{Tr}(\mathbf{A}\mathbf{U}\mathbf{B}\mathbf{U})\mathbf{V}\otimes\mathbf{V} + \\ & + \mathbf{V}\otimes\mathbf{M}'\mathbf{A}'\mathbf{U}\mathbf{B}'\mathbf{M} + \mathbf{M}'\mathbf{A}\mathbf{U}\mathbf{B}\mathbf{M}\otimes\mathbf{V}].\end{aligned} \tag{4.3}$$

For simplicity, let us denote

$$\begin{aligned}\mathbf{S}^{\mathrm{O}} &= \frac{1}{n-r(\mathbf{X})}\mathbf{Y}'\mathbf{M}_{\mathbf{X}}\mathbf{Y} + \frac{1}{n}\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{Y}\mathbf{M}_{\mathbf{Z}'} - \frac{1}{n-r(\mathbf{X})}\mathbf{M}_{\mathbf{Z}'}\mathbf{Y}'\mathbf{M}_{\mathbf{X}}\mathbf{Y}\mathbf{M}_{\mathbf{Z}'} \\ &= \mathbf{S}+\mathbf{Q}-\mathbf{W}.\end{aligned}$$

Then on the basis of theorem (4.3) and relation (3.2) we can write

$$\begin{aligned}\operatorname{Var}(\operatorname{vec}\mathbf{S}) &= \frac{1}{n-r(\mathbf{X})}\left(\mathbf{I}_{p^2} + \mathbf{K}_{pp}\right)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}), \\ \operatorname{Var}(\operatorname{vec}\mathbf{Q}) &= \frac{1}{n}\left(\mathbf{I}_{p^2} + \mathbf{K}_{pp}\right)(\mathbf{M}_{\mathbf{Z}'}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{Z}'}\otimes\mathbf{M}_{\mathbf{Z}'}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{Z}'}), \\ \operatorname{Var}(\operatorname{vec}\mathbf{W}) &= \frac{1}{n-r(\mathbf{X})}\left(\mathbf{I}_{p^2} + \mathbf{K}_{pp}\right)(\mathbf{M}_{\mathbf{Z}'}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{Z}'}\otimes\mathbf{M}_{\mathbf{Z}'}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{Z}'})\end{aligned}$$

and

$$\begin{aligned}\operatorname{Cov}(\operatorname{vec}\mathbf{S}, \operatorname{vec}\mathbf{Q}) &= \frac{1}{n}(\mathbf{M}_{\mathbf{Z}'}\otimes\mathbf{M}_{\mathbf{Z}'})\left(\mathbf{I}_{p^2} + \mathbf{K}_{pp}\right)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}), \\ \operatorname{Cov}(\operatorname{vec}\mathbf{S}, \operatorname{vec}\mathbf{W}) &= \frac{1}{n-r(\mathbf{X})}(\mathbf{M}_{\mathbf{Z}'}\otimes\mathbf{M}_{\mathbf{Z}'})\left(\mathbf{I}_{p^2} + \mathbf{K}_{pp}\right)(\boldsymbol{\Sigma}\otimes\boldsymbol{\Sigma}), \\ \operatorname{Cov}(\operatorname{vec}\mathbf{Q}, \operatorname{vec}\mathbf{W}) &= \frac{1}{n}\left(\mathbf{I}_{p^2} + \mathbf{K}_{pp}\right)(\mathbf{M}_{\mathbf{Z}'}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{Z}'}\otimes\mathbf{M}_{\mathbf{Z}'}\boldsymbol{\Sigma}\mathbf{M}_{\mathbf{Z}'}).\end{aligned}$$

So for variance of vec operator of matrix $\mathbf{S}^{\mathrm{O}}$ applies

$$\mathrm{Var}\left(\mathrm{vec}\mathbf{S}^{\mathrm{O}}\right)=\left(\frac{1}{n-r(\mathbf{X})}\mathbf{I}_{p^2}-\frac{2r(\mathbf{X})}{n(n-r(\mathbf{X}))}(\mathbf{M}_{\mathbf{Z}'}\otimes\mathbf{M}_{\mathbf{Z}'})\right)\left(\mathbf{I}_{p^2}+\mathbf{K}_{pp}\right)(\mathbf{\Sigma}\otimes\mathbf{\Sigma})+ \\ +\frac{r(\mathbf{X})}{n(n-r(\mathbf{X}))}\left(\mathbf{I}_{p^2}+\mathbf{K}_{pp}\right)(\mathbf{M}_{\mathbf{Z}'}\mathbf{\Sigma}\mathbf{M}_{\mathbf{Z}'}\otimes\mathbf{M}_{\mathbf{Z}'}\mathbf{\Sigma}\mathbf{M}_{\mathbf{Z}'}).$$

It is easy to see that in the case of uniform correlation structure it holds

$$\mathrm{Tr}\left((\mathbf{M}_{\mathbf{Z}'}\mathbf{\Sigma})^2\right)=\sigma^4(1-\rho)^2(p-r(\mathbf{Z}))^2 \quad \text{and} \quad \mathrm{Tr}\left(\mathbf{M}_{\mathbf{Z}'}\mathbf{\Sigma}^2\right)=\sigma^4(1-\rho)^2(p-r(\mathbf{Z})).$$

Using these relations we can derive the following

$$\begin{aligned}\mathrm{Var}\left(\mathrm{Tr}\,\mathbf{S}^{\mathrm{O}}\right)&=\left(\mathrm{vec}\mathbf{I}_p\right)\mathrm{Var}\left(\mathrm{vec}\mathbf{S}^{\mathrm{O}}\right)\left(\mathrm{vec}\,\mathbf{I}_p\right)\\ &=\frac{2}{n-r(\mathbf{X})}\left[\mathrm{Tr}\left(\mathbf{\Sigma}^2\right)-\frac{2r(\mathbf{X})}{n}\mathrm{Tr}\left(\mathbf{M}_{\mathbf{Z}'}\mathbf{\Sigma}^2\mathbf{M}_{\mathbf{Z}'}\right)+\frac{r(\mathbf{X})}{n}\mathrm{Tr}\left((\mathbf{M}_{\mathbf{Z}'}\mathbf{\Sigma}\mathbf{M}_{\mathbf{Z}'})^2\right)\right]\\ &=\frac{2\sigma^4}{n-r(\mathbf{X})}\left[p\left(1+(p-1)\rho^2\right)+\frac{r(\mathbf{X})}{n}(1-\rho)^2(p-r(\mathbf{Z}))(p-r(\mathbf{Z})-2)\right],\end{aligned}$$

$$\begin{aligned}\mathrm{Var}\left(\mathbf{1}'\mathbf{S}^{\mathrm{O}}\mathbf{1}\right)&=\left(\mathrm{vec}\mathbf{J}_p\right)\mathrm{Var}\left(\mathrm{vec}\mathbf{S}^{\mathrm{O}}\right)\left(\mathrm{vec}\mathbf{J}_p\right)=\frac{2}{n-r(\mathbf{X})}(\mathbf{1}'\mathbf{\Sigma}\mathbf{1})^2\\ &=\frac{2}{n-r(\mathbf{X})}p^2\sigma^4\left(1+(p-1)\rho\right)^2,\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}\left(\mathbf{1}'\mathbf{S}^{\mathrm{O}}\mathbf{1},\mathrm{Tr}\mathbf{S}^{\mathrm{O}}\right)&=\left(\mathrm{vec}\mathbf{J}_p\right)\mathrm{Var}\left(\mathrm{vec}\mathbf{S}^{\mathrm{O}}\right)\left(\mathrm{vec}\,\mathbf{I}_p\right)=\frac{2}{n-r(\mathbf{X})}\mathbf{1}'\mathbf{\Sigma}^2\mathbf{1}\\ &=\frac{2}{n-r(\mathbf{X})}p\sigma^4\left(1+(p-1)\rho\right)^2.\end{aligned}$$

The estimator of σ^2 is unbiased seeing that $\mathrm{E}\left(\hat{\sigma}_0^2\right)=\sigma^2$, so mean square error of this estimator is equal to its variance and applies

$$\mathrm{MSE}\left(\hat{\sigma}^2_{\mathrm{O}}\right)=\mathrm{Var}\left(\hat{\sigma}^2_{\mathrm{O}}\right)=\frac{2\sigma^4}{n-r(\mathbf{X})}\cdot\frac{1+(p-1)\rho^2}{p}[1+ \\ +\frac{r(\mathbf{X})(1-\rho)^2(p-r(\mathbf{Z}))(p-r(\mathbf{Z})-2)}{np\left(1+(p-1)\rho^2\right)}].$$

The estimator of parameter ρ is biased as usual. To derive properties of this estimator we will use the Taylor expansion. Using this we can find expansions of mean and variance of ratio of two variables in the form

$$\mathrm{E}\left(\frac{G}{H}\right) = \frac{\mathrm{E}(G)}{E(H)} - \frac{\mathrm{Cov}(G,H)}{(\mathrm{E}(H))^2} + \frac{\mathrm{E}(G)}{(\mathrm{E}(H))^3}\mathrm{Var}(H) + O\left(n^{-2}\right), \tag{4.4}$$

$$\mathrm{Var}\left(\frac{G}{H}\right) = \frac{\mathrm{Var}(G)}{(\mathrm{E}(H))^2} - \frac{2\mathrm{E}(G)}{(\mathrm{E}(H))^3}\mathrm{Cov}(G,H) + \frac{(\mathrm{E}(G))^2}{(\mathrm{E}(H))^4}\mathrm{Var}(H) + O\left(n^{-2}\right). \tag{4.5}$$

Using relations

$$\mathrm{E}(\hat{\rho}_{\mathrm{O}}) = \frac{1}{p-1}\left(\mathrm{E}\left(\frac{\mathbf{1}'\mathbf{S}^{\mathrm{O}}\mathbf{1}}{\mathrm{Tr}\left(\mathbf{S}^{\mathrm{O}}\right)}\right) - 1\right), \quad \mathrm{Var}(\hat{\rho}_{\mathrm{O}}) = \frac{1}{(p-1)^2}\mathrm{Var}\left(\frac{\mathbf{1}'\mathbf{S}^{\mathrm{O}}\mathbf{1}}{\mathrm{Tr}\left(\mathbf{S}^{\mathrm{O}}\right)}\right)$$

and expansions (4.4) and (4.5) we can derive that

$$\mathrm{E}(\hat{\rho}_{\mathrm{O}}) = \rho - \frac{2}{n-r(\mathbf{X})}\rho(1-\rho)\frac{1+(p-1)\rho}{p}[1 - \frac{1-\rho}{\rho} \times$$
$$\times \frac{r(\mathbf{X})}{p(p-1)}(p-r(\mathbf{Z}))(p-r(\mathbf{Z})-2)] + O(n^{-2}),$$

$$\mathrm{Var}(\hat{\rho}_{\mathrm{O}}) = \frac{2}{n-r(\mathbf{X})}\left(\frac{1+(p-1)\rho}{p-1}\right)^2\left[1 - \frac{1+(p-1)(2-\rho)\rho}{p}\right] + O\left(n^{-2}\right).$$

Similarly, using the Taylor expansion, we get mean square errors in the following form

$$\mathrm{MSE}(\hat{\rho}_{\mathrm{O}}) = \frac{2}{n-r(\mathbf{X})}(1-\rho)^2\frac{(1+(p-1)\rho)^2}{p(p-1)}[1 + \frac{r(\mathbf{X})}{np(p-1)} \times$$
$$\times (p-r(\mathbf{Z}))(p-r(\mathbf{Z})-2)] + O\left(n^{-2}\right),$$

$$\mathrm{MSE}\left(\hat{\sigma}^2_O, \hat{\rho}_O\right) = \frac{2\sigma^2}{n-r(X)}\rho(1-\rho)\frac{1+(p-1)\rho}{p}\left[1-\frac{1-\rho}{\rho}\times \right.$$
$$\left.\times\frac{r(\mathbf{X})}{np(p-1)}(p-r(\mathbf{Z}))(p-r(\mathbf{Z})-2)\right]+O\left(n^{-2}\right).$$

5. Comparisons and simulations

By the same way as described above we can derive that for Z-, M- and B-estimators of unknown parameters in growth curve model with the uniform correlation structure the following holds

$$\mathrm{E}\,\hat{\sigma}^2_Z = \sigma^2,$$
$$\mathrm{MSE}\,\hat{\sigma}^2_Z = \mathrm{Var}\,\hat{\sigma}^2_Z = \frac{2\sigma^4}{n-r(\mathbf{X})}\frac{1+(p-1)\rho^2}{p},$$
$$\mathrm{E}\hat{\rho}_Z = \rho - \frac{2}{n-r(\mathbf{X})}\rho(1-\rho)\frac{1+(p-1)\rho}{p}+O\left(n^{-2}\right),$$
$$\mathrm{MSE}\hat{\rho}_Z = \frac{2}{n-r(\mathbf{X})}\cdot\frac{(1-\rho)^2\left[1+(p-1)\rho\right]^2}{p(p-1)}+O(n^{-2}),$$
$$\mathrm{MSE}\left(\hat{\sigma}^2_Z, \hat{\rho}_Z\right) = \mathrm{Cov}\left(\hat{\sigma}^2_Z, \hat{\rho}_Z\right) = \frac{2\sigma^2}{n-r(\mathbf{X})}\rho(1-\rho)\frac{1+(p-1)\rho}{p}+O(n^{-2}),$$

$$\mathrm{E}\,\hat{\sigma}^2_M = \sigma^2\left[1-\frac{r(\mathbf{X})(r(\mathbf{Z})+\rho(p-r(\mathbf{Z})))}{np}\right],$$
$$\mathrm{Var}\hat{\sigma}^2_M = \sigma^4\frac{2}{np}\left[1+(p-1)\rho^2\right]+$$
$$-\sigma^4\frac{2r(\mathbf{X})}{n^2p^2}\left[p+p(p-1)\rho^2-(p-r(\mathbf{Z}))(1-\rho)^2\right],$$
$$\mathrm{MSE}\,\hat{\sigma}^2_M = \mathrm{Var}\hat{\sigma}^2_M + \sigma^4\frac{(r(\mathbf{X}))^2}{n^2p^2}\left[r(\mathbf{Z})+\rho(p-r(\mathbf{Z}))\right]^2,$$

$$
\begin{aligned}
E\hat{\rho}_M &= \rho - \frac{1+(p-1)\rho}{(p-1)\left[np - r(\mathbf{X})((p-r(\mathbf{Z}))\rho + r(\mathbf{Z}))\right]^3} \times \\
&\times \ \{n^2\left[p^2(p-r(\mathbf{Z}))(1-\rho)r(\mathbf{X}) + 2p^2(p-1)\rho(1-\rho)\right] + \\
&- \ 2n[(1-\rho)p(p-r(\mathbf{Z}))((p-r(\mathbf{Z}))\rho + r(\mathbf{Z}))r(\mathbf{X})^2 + \\
&+ \ p^2(p+r(\mathbf{Z})-2)(1-\rho)\rho r(\boldsymbol{X})] + \\
&+ \ (1-\rho)(p-r(\mathbf{Z}))((p-r(\mathbf{Z}))\rho + r(\mathbf{Z}))^2 r(\mathbf{X})^3 + \\
&+ \ 2p^2(r(\mathbf{Z})-1)(1-\rho)\rho r(\mathbf{X})^2\} + O(n^{-2})
\end{aligned}
$$

$$
\mathrm{MSE}\hat{\rho}_{\mathrm{M}} = \frac{2n^3p^3(p-1)(1-\rho)^2[1+(p-1)\rho]^2}{\left[np - r(\mathbf{X})((p-r(\mathbf{Z}))\rho + r(\mathbf{Z}))\right]^4(p-1)^2} + O\left(n^{-2}\right),
$$

$$
\mathrm{MSE}\left(\hat{\sigma}^2{}_{\mathrm{M}}, \hat{\rho}_M\right) = \sigma^2 \frac{2n^2p^2\rho(1-\rho)(1+(p-1)\rho)}{\left[np - r(\mathbf{X})((p-r(\mathbf{Z}))\rho + r(\mathbf{Z}))\right]^3} + O(n^{-2}),
$$

$$
\mathrm{E}\ \hat{\sigma}^2{}_{\mathrm{B}} = \sigma^2,
$$

$$
\begin{aligned}
\mathrm{MSE}\ \hat{\sigma}^2{}_{\mathrm{B}} &= \mathrm{Var}\hat{\sigma}^2{}_{\mathrm{B}} = \frac{2\sigma^4}{n-r(\mathbf{X})} \cdot \frac{1}{p^2\left[n(p-1) - r(\mathbf{X})(r(\mathbf{Z})-1)\right]} \times \\
&\times \{np(p-1)\left[1+(p-1)\rho^2\right] - r(\mathbf{X})[(r(\mathbf{Z})-1)(2p - r(\mathbf{Z}) + \\
&+ (p-1)^2\rho^2) + ((p-1)\rho + (p-r(\mathbf{Z})))^2]\},
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{E}\hat{\rho}_{\mathrm{B}} = \ & \rho - \frac{2(1+(p-1)\rho)(1-\rho)\left[np(p-1)\rho - r(\mathbf{X})(pr(\mathbf{Z})\rho - p - r(\mathbf{Z})\rho + r(\mathbf{Z}))\right]}{p^2\left[n^2(p-1) - n(p+r(\mathbf{Z})-2)r(\mathbf{X}) + (r(\mathbf{Z})-1)r(\mathbf{X})^2\right]} + \\
& + O(n^{-2}),
\end{aligned}
$$

$$
\mathrm{MSE}\hat{\rho}_{\mathrm{B}} = \frac{2}{n-r(\mathbf{X})} \cdot \frac{(1-\rho)^2\left[1+(p-1)\rho\right]^2\left(np - r(\mathbf{X})r(\mathbf{Z})\right)}{p^2\left[n(p-1) - r(\mathbf{X})(r(\mathbf{Z})-1)\right]} + O(n^{-2})
$$

$$
\begin{aligned}
\mathrm{MSE}\left(\hat{\sigma}^2{}_{\mathrm{B}}, \hat{\rho}_{\mathrm{B}}\right) = \ & \mathrm{Cov}\left(\hat{\sigma}^2{}_{\mathrm{B}}, \hat{\rho}_{\mathrm{B}}\right) = \frac{2\sigma^2}{n-r(\mathbf{X})} \frac{(1-\rho)\left[1+(p-1)\rho\right]}{p} \times \\
& \times \frac{np(p-1)\rho - r(\mathbf{X})\left[r(\mathbf{Z})(p-1)\rho - (p-r(\mathbf{Z}))\right]}{p\left[n(p-1) - r(\mathbf{X})(r(\mathbf{Z})-1)\right]} + O(n^{-2}).
\end{aligned}
$$

These estimators of unknown parameters in the growth curve model with uniform correlation structure compares Žežula (2006). He shows that $\hat{\rho}_Z$ and $\hat{\rho}_B$ are less biased than $\hat{\rho}_M$. On the ground of negative bias of estimators $\hat{\rho}_Z$ and $\hat{\rho}_B$ and from equality

$$\mathrm{E}\,\hat{\rho}_B - \mathrm{E}\,\hat{\rho}_Z = -\frac{2(1-\rho)^2(1+(p-1)\rho)(p-r(\mathbf{Z}))r(\mathbf{X})}{p^2(n-r(\mathbf{X}))(np-n+r(\mathbf{X})-r(\mathbf{X})r(\mathbf{Z}))} + O\left(n^{-2}\right)$$

it is implied that $\hat{\rho}_Z$ is always less biased than $\hat{\rho}_B$. Some simulations of this difference as a function of parameter ρ are depicted in the Fig. 1.

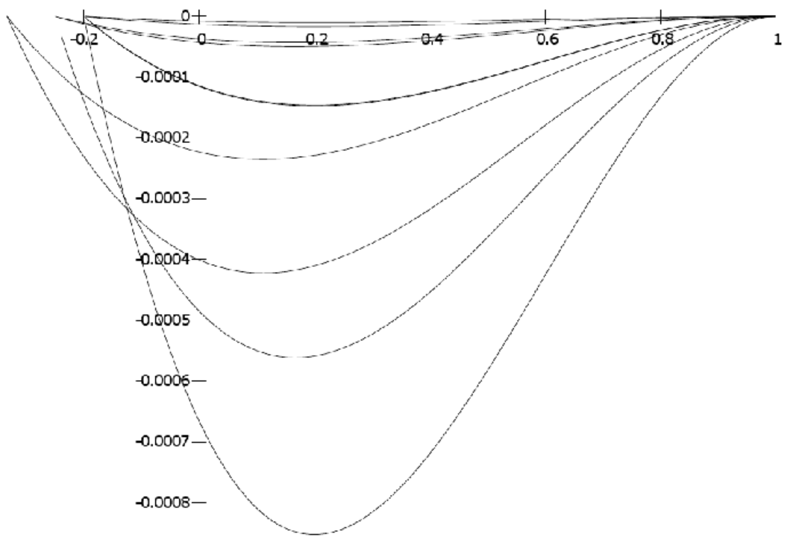


Fig. 1. $\mathrm{E}\,\hat{\rho}_B - \mathrm{E}\hat{\rho}_Z$

Now we look at $\mathrm{O}-$estimators. With an accuracy of n^{-1} we compare these estimators and show some simulations. Variance of Z-estimator of parameter ρ is the same as the variance of $\mathrm{O}-$estimator of this parameter and applies

$$\mathrm{Var}\left(\hat{\rho}_Z\right) = \mathrm{Var}\left(\hat{\rho}_O\right) = \frac{2}{n-r(\mathbf{X})}\left(\frac{1+(p-1)\rho}{p-1}\right)^2\left[1-\frac{1+(p-1)(2-\rho)\rho}{p}\right] + \\ + O\left(n^{-2}\right).$$

Note that this term is nonnegative for all admissible values of p and ρ. Next, if $p = r(\mathbf{Z})$ or $p = r(\mathbf{Z}) + 2$ then these estimators have the same properties as Z-estimators, so

$\mathrm{MSE}\hat{\sigma}^2_{\mathrm{O}} = \mathrm{Var}\,\hat{\sigma}^2_{\mathrm{O}} = \mathrm{MSE}\,\hat{\sigma}^2_{\mathrm{Z}} = \mathrm{Var}\hat{\sigma}^2_{\mathrm{Z}}$,

$\mathrm{MSE}\left(\hat{\sigma}^2_{\mathrm{O}}, \hat{\rho}_{\mathrm{O}}\right) = \mathrm{MSE}\left(\hat{\sigma}^2_{\mathrm{Z}}, \hat{\rho}_{\mathrm{Z}}\right)$, $\mathrm{MSE}\,\hat{\rho}_{\mathrm{O}} = \mathrm{MSE}\hat{\rho}^2_{\mathrm{Z}}$ and $\mathrm{E}\hat{\rho}_{\mathrm{O}} = \mathrm{E}\,\hat{\rho}_{\mathrm{Z}}$.

This situation occurs for example when we use Potthoff's and Roy's dental data. Estimator $\hat{\rho}_{\mathrm{O}}$ is biased, in some cases positively biased and in some cases negatively biased. For differences between Z- and B-estimators it holds

$$\mathrm{E}\,\hat{\rho}_{\mathrm{Z}} - \mathrm{E}\hat{\rho}_{\mathrm{O}} = \frac{2r(\mathbf{X})\left[1+(p-1)\rho\right](1-\rho)^2(p-r(\mathbf{Z}))(p-r(\mathbf{Z})-2)}{p^2(p-1)(n-r(\mathbf{X}))} + O\left(n^{-2}\right),$$

$$\mathrm{E}\,\hat{\rho}_{\mathrm{B}} - \mathrm{E}\,\hat{\rho}_{\mathrm{O}} = -\frac{2(1-\rho)^2(1+(p-1)\rho)(p-r(\mathbf{Z}))r(\mathbf{X})}{p^2(n-r(\mathbf{X}))(np-n+r(\mathbf{X})-r(\mathbf{X})r(\mathbf{Z}))} \times \left[1+(p-r(\mathbf{Z})-2)\left(n+\frac{r(\mathbf{X})(1-r(\mathbf{Z}))}{p-1}\right)\right] + O\left(n^{-2}\right)$$

and some simulations of these differences as functions of ρ are shown in the figures 2 and 3.

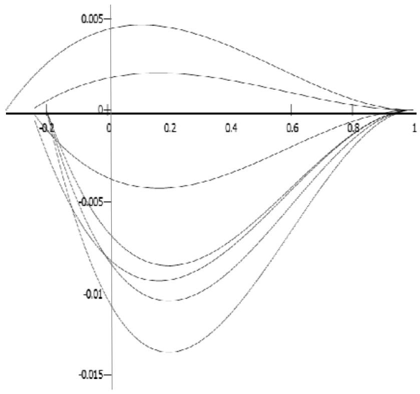


Fig. 2. $\mathrm{E}\,\hat{\rho}_{\mathrm{Z}} - \mathrm{E}\hat{\rho}_{\mathrm{O}}$

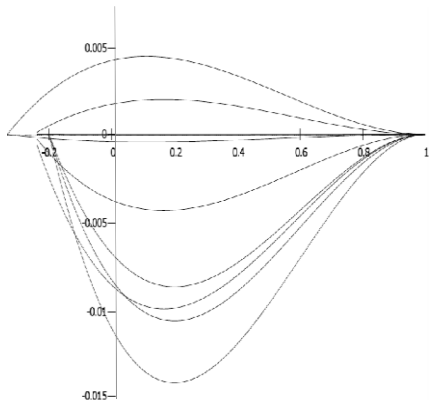


Fig. 3. $\mathrm{E}\,\hat{\rho}_{\mathrm{B}} - \mathrm{E}\hat{\rho}_{\mathrm{O}}$

Estimators $\hat{\sigma}^2_O$, $\hat{\sigma}^2_Z$ and $\hat{\sigma}^2_B$ are unbiased, while $\hat{\sigma}^2_M$ is biased and always less then σ^2. The same data as above we use to show ratios $\frac{MSE\,\hat{\sigma}^2_O}{MSE\hat{\sigma}^2_Z}-1$, $\frac{MSE\hat{\sigma}^2_O}{MSE\hat{\sigma}^2_B}-1$ and $\frac{MSE\hat{\sigma}^2_O}{MSE\hat{\sigma}^2_M}-1$ as functions of the parameter ρ as above (figures 4–6).

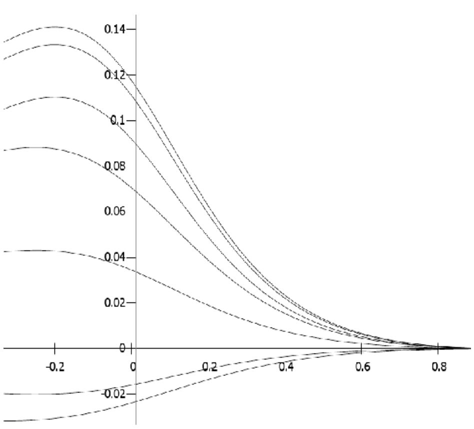


Fig. 4. $\frac{MSE\hat{\sigma}^2_O}{MSE\hat{\sigma}^2_Z}-1$

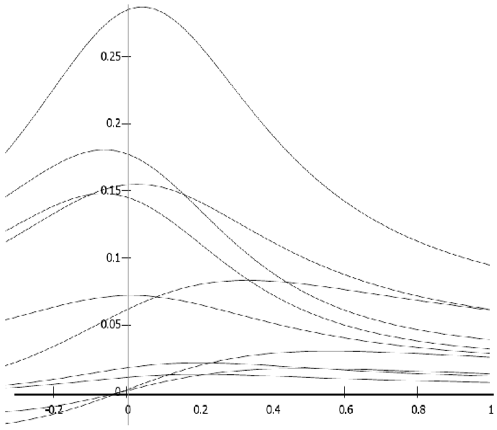


Fig. 5. $\frac{MSE\hat{\sigma}^2_O}{MSE\hat{\sigma}^2_B}-1$

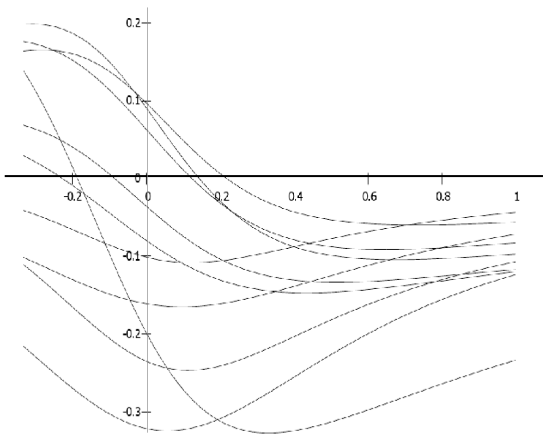
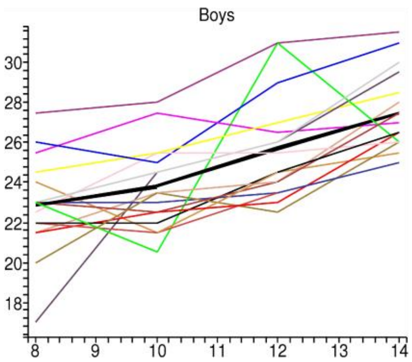


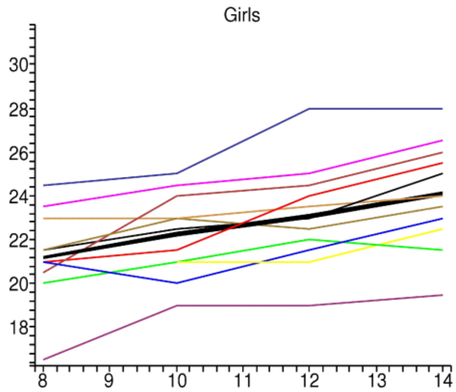
Fig. 6. $\frac{MSE\,\hat{\sigma}^2_O}{MSE\hat{\sigma}^2_M}-1$

The simulations are made for small values of time points (3-10) and bigger number of subjects (10-100). On the basis of these pictures we can see that none of considered estimators has uniformly minimal mean square error. But estimators $\hat{\sigma}^2{}_Z$ and $\hat{\sigma}^2{}_B$ seem to be better than estimator $\hat{\sigma}^2{}_O$ in most cases. We can consider $\hat{\sigma}^2{}_B$ better than $\hat{\sigma}^2{}_O$ for positive values of parameter ρ. On the other hand estimator $\hat{\sigma}^2{}_O$ seems to be better than estimator $\hat{\sigma}^2{}_M$ for positive values of parameter ρ tend to one.

6. Example - Potthoff's and Roy's dental data

The growth curve model was considered in 1964 by Richard F. Potthoff and Samarendra Nath Roy. They tried to answer the question, if the distance between the center of the pituitary to the pterygomaxillary fissure is the same for boys and girls and if its growth rate is the same for both groups. The observations were collected from 11 girls and 16 boys at four different ages, specifically 8, 10, 12, and 14 years. The obtained data (see Potthoff and Roy, 1964) are displayed in the next figures.





Thick lines in both pictures represent averages in groups. On the basis of this we can consider linear trend for boys and for girls. ANOVA matrix $\mathbf{X}$, matrix $\mathbf{B}$ of unknown parameters and matrix $\mathbf{Z}$ of regression constants for this example are of the form:

$$\mathbf{X} = \begin{pmatrix} \mathbf{1}_{11} & \mathbf{0}_{16} \\ \mathbf{0}_{11} & \mathbf{1}_{16} \end{pmatrix}', \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{Z} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 8 & 10 & 12 & 14 \end{pmatrix}.$$

We dealt with three estimators of variance matrix. With UMVUIE $\mathbf{S}$, maximum likelihood estimator $\mathbf{S}^{\mathbf{M}}$ and outer product estimator $\mathbf{S}^{\mathrm{O}}$. For dental data these matrices are

$$\mathbf{S} = \begin{pmatrix} 5.4155 & 2.7168 & 3.9102 & 2.7102 \\ 2.7168 & 4.1848 & 2.9272 & 3.3172 \\ 3.9102 & 2.9272 & 6.4557 & 4.1307 \\ 2.7102 & 3.3172 & 4.1307 & 4.9857 \end{pmatrix},$$

$$\mathbf{S}^{\mathbf{M}} = \begin{pmatrix} 5.0545 & 2.4578 & 3.6157 & 2.5320 \\ 2.4578 & 3.9582 & 2.7170 & 3.0392 \\ 3.6157 & 2.7170 & 5.9788 & 3.8217 \\ 2.5320 & 3.0392 & 3.8217 & 4.6292 \end{pmatrix}$$

and

$$\mathbf{S}^{\mathrm{O}} = \begin{pmatrix} 5.4262 & 2.7080 & 3.8958 & 2.7228 \\ 2.7080 & 4.1624 & 2.9985 & 3.2771 \\ 3.8958 & 2.9985 & 6.3563 & 4.1732 \\ 2.7228 & 3.2771 & 4.1732 & 4.9708 \end{pmatrix}.$$

Let us consider uniform correlation structure. Then various estimators of unknown parameters derived in this article and residual sums of squares are in the following table.

estimator of parameter σ^2	estimator of parameter ρ	RSS
$\hat{\sigma}^2_{\mathrm{Z}} = 5.2604$	$\hat{\rho}_{\mathrm{Z}} = 0.6245$	$S_e = 101.88$
$\hat{\sigma}^2_{\mathrm{M}} = 4.9052$	$\hat{\rho}_{\mathrm{M}} = 0.6178$	$S_e = 108.00$
$\hat{\sigma}^2_{\mathrm{B}} = 5.2207$	$\hat{\rho}_{\mathrm{B}} = 0.6318$	$S_e = 104.00$
$\hat{\sigma}^2_{\mathrm{O}} = 5.2289$	$\hat{\rho}_{\mathrm{O}} = 0.6303$	$S_e = 103.55$

These estimators are very similar except for the maximum likelihood estimators, which is reflected also by residual sums of squares.

Acknowledgement

This work was supported by grant VEGA 1/0410/11 and VVGS-PF-2012-45.

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