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REGULAR D−OPTIMAL WEIGHING DESIGNS WITH NEGATIVE CORRELATED ERRORS: CONSTRUCTION

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Summary

The chemical balance weighing designs satisfying the criterion of D-optimality under assumption that the measurements are negative correlated and they have the same variances are considered. We present new method of construction of D-optimal designs based on the set of the incidence matrices of the balanced bipartite weighing designs. Besides, we give an examples of the design matrix

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1. Introduction

Suppose, we determine unknown measurements of p objects using n weighing operations. The results of this experiment can be described by the model $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$, where

(i) $\mathbf{y}$ is an $n \times 1$ random vector of the observations,

(ii) $\mathbf{w} = (w_1, w_2, ..., w_p)'$ is a vector representing unknown measurements of objects,

(iii) $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$, where $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ denotes the class of $n \times p$ matrices having entries $x_{ij} = -1, 1$ or 0,

(iv) m is the maximum number of elements equal to 1 and -1 in each column of the matrix $\mathbf{X}$,

(v) $\mathbf{e}$ is the $n\times 1$ random vector of errors. We shall make two standing assumptions on the maps of our considerations: $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where

$$\mathbf{G} = g\left((1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n'\right),\ g > 0,\ \frac{-1}{n-1} < \rho < 0\,. \tag{1.1}$$

In the case where $\rho \in \left(\frac{-1}{n-1}, 0\right)$ and $g > 0$, $\mathbf{G}$ is positive definite. So, if the matrix $\mathbf{X}$ is of full column rank, then using the weighted lest squares method we obtain the best linear unbiased estimator $\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ with the variance matrix $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$. The matrix $\mathbf{M} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix of the design $\mathbf{X}$.

The optimality problem is concerned with efficient estimation in some sense by a proper choice of the design matrix $\mathbf{X}$. There are many possible optimality criteria. One of them is D-optimality. This optimality criterion minimizes the determinant of the matrix $\mathbf{M}^{-1}$. The design $\mathbf{X}_D$ is called D-optimal in the class of all possible design matrices $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ if $\det(\mathbf{X}_D) = \min\left(\det\left(\mathbf{M}^{-1}\right): \mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}\right)$. If $\det(\mathbf{X}_D)$ attains the lowest bound then the design is called regular D-optimal. In other cases it is called D-optimal. The research related to D-optimality was presented for instance in Raghavarao (1971), Shah and Sinha (1989). Some results concerned on the regular D-optimal chemical balance weighing designs are given in literature including Gail and Kiefer (1982), Jacroux et al. (1983), Chadjiconstantinidis and Chadjipadelis (1994), Koukouvinos (1996), Abrego et al. (2003), Masaro and Wong (2008) ones.

Ceranka and Graczyk (2014a) gave the definition of the regular D-optimal chemical balance weighing design and the conditions determining regular D-optimal design for $\rho \in \left(\frac{-1}{n-1}, 0 \right)$.

Definition 1.1. Let $\rho \in \left(\frac{-1}{n-1}, 0 \right)$. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is regular D-optimal if

$\det\left(\mathbf{M}^{-1}\right) = \left(g(1-\rho)\left(m - \frac{\rho(m-2u)^2}{1+\rho(n-1)} \right)^{-1} \right)^p$, where $u = \min\{u_1, u_2, ..., u_p\}$, u_j represent the number of elements equal to -1 in j th column of $\mathbf{X}$, $j = 1,2,..., p$.

Theorem 1.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is regular D-optimal if and only if

$$\mathbf{X}'\mathbf{X} = m\mathbf{I}_p - \frac{\rho(m-2u)^2}{1+\rho(n-1)}\left(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p'\right) \quad (1.2)$$

$$\mathbf{X}'\mathbf{1}_n = \mathbf{z}_p, \quad (1.3)$$

where $\mathbf{z}_p$ is $p \times 1$ vector for which the j th element is equal to $-(m-2u)$ or $m-2u$, $j = 1,2,..., p$.

2. The construction

For $\mathbf{X} \in \mathbf{\Pi}_{n\times p}\{-1, 1\}$, where $\mathbf{\Pi}_{n\times p}\{-1, 1\}$ denotes the class of $n \times p$ matrices having entries $x_{ij} = -1$ or 1, some constructions of regular D-optimal designs were given in Masaro and Wong (2008). In more extensive class of the design matrices $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$, the problem of optimality in weighing designs was considered in Ceranka and Graczyk (2014a).

In this Section we present new method of construction of the regular D-optimal chemical balance weighing design based on the set of the incidence matrices of balanced bipartite weighing designs. This construction was motivated by the properties of such designs. For a deeper discussion of such designs we refer the reader to Huang (1976). Let $\mathbf{N}_h^*$ $(h=1,2,...,t)$, be the incidence matrix of balanced bipartite weighing design with the parameters v, b_h, r_h, k_{1h}, k_{2h}, λ_{1h}, λ_{2h}. From $\mathbf{N}_h^*$ we form the matrix $\mathbf{N}_h$ by replacing k_{1h} unities equal to +1 of each column which correspond to the elements belonging to the first subblock by -1. Then each column of the matrix $\mathbf{N}_h$ will contain k_{1h} elements equal to -1, k_{2h} elements equal to 1 and $v-k_{1h}-k_{2h}$ elements equal to 0. Let $\mathbf{X}\in\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ be the design matrix of the chemical balance weighing design in the form

$$\mathbf{X}=\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2 & \cdots & \mathbf{N}_t\end{bmatrix}'. \qquad (2.1)$$

Therefore in such design we determine unknown measurements of $p=v$ objects in $n=\sum_{h=1}^{t} b_h$ measurement operations, each object is weighted $m=\sum_{h=1}^{t} r_h$ times.

The result given by Ceranka and Graczyk (2014b) will be needed for next considerations.

Lemma 2.1. The chemical balance weighing design with the design matrix $\mathbf{X}\in\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) is nonsingular if and only if

$$k_{1h}\neq k_{2h} \qquad (2.2)$$

for at least one $h=1,2,...,t$.

Without loss of generality, we can assume that $k_{1h}\neq k_{2h}$ for each h. Then $r_{1h}=\frac{\lambda_{1h}(v-1)}{2k_{2h}}$ and $r_{2h}=\frac{\lambda_{1h}(v-1)}{2k_{1h}}$, $h=1,2,...,t$, are the numbers of elements equal to -1 and 1 in each column of $\mathbf{N}_h'$. We will work under this assumption.

Ceranka and Graczyk (2014a) gave the condition determining regular D-optimal design for the case $\rho \in \left(\frac{-1}{n-1}, 0 \right)$.

From Theorem 1.1 we can see that the optimality conditions are depended on the parameter ρ in (1.1). This implies that the methods of construction of the design $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}\{-1, 0, 1\}$ are depended on ρ, either. Therefore we will formulate our main result that provides criteria of the regular D-optimal designs.

Theorem 2.1. Let $k_{1h} \neq k_{2h}$ for each $h = 1,2,...,t$. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}\{-1, 0, 1\}$ in the form (2.1) with the covariance matrix of errors $\sigma^2 \mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is regular D-optimal if and only if the following conditions are simultaneously satisfied

$$\rho = \frac{\sum_{h=1}^{t} (\lambda_{2h} - \lambda_{1h})}{\left(\sum_{h=1}^{t} (r_{2h} - r_{1h}) \right)^2 - \left(\sum_{h=1}^{t} b_h - 1 \right) \left(\sum_{h=1}^{t} (\lambda_{2h} - \lambda_{1h}) \right)}, \tag{2.3}$$

$$\sum_{h=1}^{t} (\lambda_{2h} - \lambda_{1h}) < 0, \tag{2.4}$$

$$\left(\sum_{h=1}^{t} (r_{2h} - r_{1h}) \right)^2 \neq 0. \tag{2.5}$$

Proof. From Theorem 1.1 it follows that if $\rho \in \left(\frac{-1}{n-1}, 0 \right)$ then chemical balance weighing design is regular D-optimal if the conditions (1.2) and (1.3) hold. From $\mathbf{X}' \mathbf{1}_n = \mathbf{z}_p$ we have $\mathbf{c}_j' \mathbf{X}' \mathbf{1}_n = m - 2u$ or $-(m-2u)$, $j = 1,2,..., p$, where $m - 2u = \sum_{h=1}^{t} (r_{2h} - r_{1h})$, $\mathbf{c}_j$ is the jth column of matrix $\mathbf{I}_p$. From the condition $\mathbf{X}' \mathbf{X} = m\mathbf{I}_p - \frac{\rho(m-2u)^2}{1+\rho(n-1)} (\mathbf{I}_p - \mathbf{1}_p \mathbf{1}_p')$ we obtain $\mathbf{c}_j' \mathbf{X}' \mathbf{X} \mathbf{c}_j = \frac{\rho(m-2u)^2}{1+\rho(n-1)}$ and consequently from (1.3) we get $\mathbf{c}_j^{'}\mathbf{X}^{'}\mathbf{X}\mathbf{c}_j = \sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h})$. Then $\sum_{h=1}^{t}(\lambda_{2h} - \lambda_{1h}) = \frac{\rho\left(\sum_{h=1}^{t}(r_{2h} - r_{1h})\right)^2}{1 + \rho\left(\sum_{h=1}^{t} b_h - 1\right)}$ and thus we have (2.3). Moreover, under the condition (2.4), the denominator of (2.3) is greater than zero, hence $\rho < 0$. Since (2.5) then $\rho \in \left(\frac{-1}{n-1}, 0\right)$. If (2.5) is not fulfilled then $\rho = -(n-1)^{-1}$ and in that case the matrix $\mathbf{G}$ is not positive definite. This completes the proof. ■

Theorem 2.2. Let $t = 1$. If for a given ρ, the parameters of the balanced bipartite weighing design are equal to

(i) $\rho = -2(108s^2 - 24s - 1)^{-1}$, $v = 6s$, $b_1 = 6s(6s-1)$, $r_1 = 3(6s-1)$, $k_1 = 1$, $k_2 = 2$, $\lambda_1 = 4$, $\lambda_2 = 2$, $s = 1,2,\ldots$,

(ii) $\rho = -(4s^2 + s - 1)^{-1}$, $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 8s$, $k_1 = 3$, $k_2 = 5$, $\lambda_1 = 15$, $\lambda_2 = 13$, $s = 4,5,\ldots$,

(iii) $\rho = -(3s^2 + s - 1)^{-1}$, $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 3s$, $k_1 = 1$, $k_2 = 2$, $\lambda_1 = 2$, $\lambda_2 = 1$, $s = 2,3,\ldots$, except the case $s = 5$,

(iv) $\rho = -(5s^2 + s - 1)^{-1}$, $v = 4s+1$, $b_1 = s(4s+1)$, $r_1 = 5s$, $k_1 = 2$, $k_2 = 3$, $\lambda_1 = 3$, $\lambda_2 = 2$, $s = 2,3,\ldots$,

(v) $\rho = -(6s^2 + s - 1)^{-1}$, $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 6s$, $k_1 = 2$, $k_2 = 4$, $\lambda_1 = 8$, $\lambda_2 = 7$, $s = 3,4,\ldots$,

(vi) $\rho = -(6s^2 - 5s)^{-1}$, $v = 2s$, $b_1 = s(2s-1)$, $r_1 = 3(2s-1)$, $k_1 = 2$, $k_2 = 4$, $\lambda_1 = 8$, $\lambda_2 = 7$, $s = 3,4,\ldots$, then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in the form (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is regular D-optimal.

Proof. The proof is immediate by observing that the parameters given in (i)-(vi) satisfy the conditions (2.3)-(2.5). ■

Theorem 2.3. Let $t = 2$. If for a given ρ, the parameters of the balanced bipartite weighing design are equal to

(i) $\rho = -3\left(306s^2 - 21s + 1\right)^{-1}$, $v = 6s$, $b_1 = 6s(6s-1)$, $r_1 = 3(6s-1)$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 4$, , $\lambda_{21} = 2$, and $v = 6s$, $b_2 = 3s(6s-1)$, $r_2 = 3(6s-1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 1,2,...$,

(ii) $\rho = -2\left((2t+1)^2(6s-1)^2 + 2(6s(6s-1)(t+1)-1)\right)^{-1}$, $v = 6s$, $b_1 = 6s(6s-1)$, $r_1 = 3(6s-1)$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 4$, $\lambda_{21} = 2$ and $v = 6s$, $b_2 = 6ts(6s-1)$, $r_2 = 4t(6s-1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 6t$, $\lambda_{22} = 6t$, $s,t = 1,2,...$,

(iii) $\rho = -2\left((t+1)^2(6s-1)^2 + 2(2s(6s-1)(t+3)-1)\right)^{-1}$, $v = 6s$, $b_1 = 6s(6s-1)$, $r_1 = 3(6s-1)$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 4$, $\lambda_{21} = 2$ and $v = 6s$, $b_2 = 2ts(6s-1)$, $r_2 = 3t(6s-1)$, $k_{12} = 3$, $k_{22} = 6$, $\lambda_{12} = 12t$, $\lambda_{22} = 12t$, $s,t = 1,2,...$,

(iv) $\rho = -\left(7s^2 + 2s - 1\right)^{-1}$, $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 8s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 2s+1$, $b_2 = s(2s+1)$, $r_2 = 3s$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$, $s = 6,7,...$,

(v) $\rho = -3\left(61s^2 + 9s - 3\right)^{-1}$, $v = 4s + 1$, $b_1 = 2s(4s + 1)$, $r_1 = 16s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 5s$, $k_{12} = 2$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 2$, $s = 2,3,\ldots$,

(vi) $\rho = -3\left(28s^2 + 6s - 3\right)^{-1}$, $v = 2s + 1$, $b_1 = s(2s + 1)$, $r_1 = 8s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 2s + 1$, $b_2 = s(2s + 1)$, $r_2 = 6s$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 4,5,\ldots$,

(vii) $\rho = -\left(61s^2 + 3s - 3\right)^{-1}$, $v = 4s + 1$, $b_1 = 2s(4s + 1)$, $r_1 = 16s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 4s + 1$, $b_2 = s(4s + 1)$, $r_2 = 5s$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$, $s = 2,3,\ldots$,

(viii) $\rho = -\left(256s^2 + 6s - 1\right)^{-1}$, $v = 10s + 1$, $b_1 = 5s(10s + 1)$, $r_1 = 40s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 10s + 1$, $b_2 = s(10s + 1)$, $r_2 = 6s$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 1$, $\lambda_{22} = 2$, $s = 1,2,\ldots$,

(ix) $\rho = -\left(29s^2 + 2s - 1\right)^{-1}$, $v = 2s + 1$, $b_1 = s(2s + 1)$, $r_1 = 8s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 2s + 1$, $b_2 = s(2s + 1)$, $r_2 = 7s$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$, $s = 4,5,\ldots$,

(x) $\rho = -\left(2s^2(t + 1)^2 + 2s^2(t + 1) + s(t + 1) - 1\right)^{-1}$, $v = 2s + 1$, $b_1 = s(2s + 1)$, $r_1 = 8s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 2s + 1$, $b_2 = ts(2s + 1)$, $r_2 = 4ts$, $k_{12} = 1, k_{22} = 3$, $\lambda_{12} = 3t$, $\lambda_{22} = 3t$, $s = 4,5,\ldots$, $t = 1,2,\ldots$,

(xi) $\rho=-2\left(s^2(3t+2)^2+2(s(t+1)(2s+1)-1)\right)^{-1}$, $v=2s+1$, $b_1=s(2s+1)$, $r_1=8s$, $k_{11}=3$, $k_{21}=5$, $\lambda_{11}=15$, $\lambda_{21}=13$ and $v=2s+1$, $b_2=ts(2s+1)$, $r_2=9ts$, $k_{12}=3$, $k_{22}=6$, $\lambda_{12}=18t$, $\lambda_{22}=18t$, $s=4,5,...$, $t=1,2,...$,

(xii) $\rho=-\left(s^2(2t+1)^2+s(2s+1)(t+1)-1\right)^{-1}$, $v=2s+1$, $b_1=s(2s+1)$, $r_1=3s$, $k_{11}=1$, $k_{21}=2$, $\lambda_{11}=2$, $\lambda_{21}=1$ and $v=2s+1$, $b_2=ts(2s+1)$, $r_2=4ts$, $k_{12}=1$, $k_{22}=3$, $\lambda_{12}=3t$, $\lambda_{22}=3t$, $s=2,3,...$, $t=1,2,...$, except the case $s=5$,

(xiii) $\rho=-\left(s^2(3t+1)^2+s(2s+1)(t+1)-1\right)^{-1}$, $v=2s+1$, $b_1=s(2s+1)$, $r_1=3s$, $k_{11}=1$, $k_{21}=2$, $\lambda_{11}=2$, $\lambda_{21}=1$ and $v=2s+1$, $b_2=ts(2s+1)$, $r_2=9ts$, $k_{12}=3$, $k_{22}=6$, $\lambda_{12}=18t$, $\lambda_{22}=18t$, $s=2,3,...$, $t=1,2,...$, except the case $s=5$,

(xiv) $\rho=-\left(s^2(4t+1)^2+s(4s+1)(2t+1)-1\right)^{-1}$, $v=4s+1$, $b_1=s(4s+1)$, $r_1=5s$, $k_{11}=2$, $k_{21}=3$, $\lambda_{11}=3$, $\lambda_{21}=2$ and $v=4s+1$, $b_2=2ts(4s+1)$, $r_2=8ts$, $k_{12}=1$, $k_{22}=3$, $\lambda_{12}=3t$, $\lambda_{22}=3t$, $s=2,3,...$, $t=1,2,...$,

(xv) $\rho=-\left(s^2(3t+1)^2+s(4s+1)(t+1)-1\right)^{-1}$, $v=4s+1$, $b_1=s(4s+1)$, $r_1=5s$, $k_{11}=2$, $k_{21}=3$, $\lambda_{11}=3$, $\lambda_{21}=2$ and $v=4s+1$, $b_2=ts(4s+1)$, $r_2=9ts$, $k_{12}=3$, $k_{22}=6$, $\lambda_{12}=9t$, $\lambda_{22}=9t$, $s=2,3,...$, $t=1,2,...$,

(xvi) $\rho = -\left(4s^2(3t+1)^2 + s(2s+1)(t+1) - 1\right)^{-1}$, $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 6s$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = 2s+1$, $b_2 = ts(2s+1)$, $r_2 = 4ts$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3t$, $\lambda_{22} = 3t$, $s = 3,4,\ldots$, $t = 1,2,\ldots$,

(xvii) $\rho = -\left(s^2(3t+2)^2 + s(2s+1)(t+1) - 1\right)^{-1}$, $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 6s$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = 2s+1$, $b_2 = ts(2s+1)$, $r_2 = 9ts$, $k_{12} = 3$, $k_{22} = 6$, $\lambda_{12} = 18t$, $\lambda_{22} = 18t$, $s = 3,4,\ldots$, $t = 1,2,\ldots$,

(xviii) $\rho = -\left((2s-1)^2(2t+1)^2 + s(2s-1)(2t+1) - 1\right)^{-1}$, $v = 2s$, $b_1 = s(2s-1)$, $r_1 = 3s(2s-1)$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = 2s$, $b_2 = 2ts(2s-1)$, $r_2 = 4t(2s-1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 6t$, $\lambda_{22} = 6t$, $s = 3,4,\ldots$, $t = 1,2,\ldots$,

then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}\{-1, 0, 1\}$ in the form (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is regular D-optimal.

Proof. One can easily check that the parameters given in (i)-(xviii) fulfilled the conditions given in Theorem 2.1. ■

3. Examples

Let us consider the experiment where unknown measurements of $p = v = 5$ objects in $n = 10$ measurement operations are estimated. We assume that each object can be measured $m = 6$ times. Suppose that the errors of measurements are negative correlated and have the same variances, so $\mathbf{G} = g\left(\frac{14}{13}\mathbf{I}_n - \frac{1}{13}\mathbf{1}_n\mathbf{1}_n'\right)$.

In this way we determine the regular D-optimal design in the class $\mathbf{\Phi}_{10\times5,6}\{-1, 0, 1\}$. The design $\mathbf{X}\in\mathbf{\Phi}_{10\times5,6}\{-1, 0, 1\}$ in the form (2.1) can be constructed from the incidence matrix of balanced bipartite weighing designs with the parameters $v=5$, $b=10$, $r=6$, $k_1=1$, $k_2=2$, $\lambda_1=2$, $\lambda_2=1$ (see, Theorem 2.3 (iii)) and

$$\mathbf{N}^* = \begin{bmatrix} 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 0 & 0 & 1_1 & 0 \\ 1_2 & 0 & 0 & 1_1 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 0 \\ 1_1 & 1_2 & 0 & 0 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_1 \\ 0 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_1 & 1_2 & 0 & 1_2 \\ 0 & 0 & 1_1 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 1_2 \end{bmatrix},$$

where 1_1 and 1_2 denote that the object exists in the first or in the second subblock, respectively, 0 if the object does not exist in the block. In each incidence matrix of balanced bipartite weighing design we replace the elements that are equal to 1 and correspond to elements belonging to the first subblock (1_1) by -1. As the next step we built design $\mathbf{\Phi}_{10\times5,6}\{-1, 0, 1\}$ in the form (2.1) for $t=1$

$$\mathbf{X}' = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 1 & 1 & 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

and $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1} = \left(\frac{2g}{13}\right)^5$.

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