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ON THE RELATIVE EFFICIENCY OF SPLIT–SPLIT–PLOT DESIGN TO SPLIT–PLOT × SPLIT–BLOCK DESIGN

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Summary

In the paper we consider two the most popular in practice designs for three-factorial experiments, i.e. split-split-plot (SSP) design and split–plot × split–block (SPSB) design. Discussed here models of observations are called randomized-derived models and are strictly connected with randomization performed for nested and crossed structures of experimental units. Statistical properties result from two different schemes of randomization applied in the experiments.

The aim of the paper is comparing both models with respect to the relative efficiency in order to decide which a design is the best for estimation and testing hypotheses for factors and interactions between them. The considerations are illustrated with an example involving a winter wheat trial.

Keywords and phrases: relative efficiency, split-split-plot design, split-plot × split-block design

Classification AMS 2010: 62K10, 62K15

1. Introduction

In design of three-factorial experiments two experimental designs are the most popular, i.e. split-split-plot (SSP) design and split–plot × split–block

(SPSB) design. In practice, they are mostly used to orthogonal (complete) experiments (here by which we mean that all treatment combinations occur once in each block). Design of such experiments, modelling and statistical analysis, for both the complete designs and incomplete designs, were discussed in many papers (e.g. Ambroży i Mejza. 2003, 2004, 2006, 2008a, 2008b, 2009, 2012a, 2012b, Mejza I., 1997a, 1997b). Models of observations which are presented in this paper are called randomized-derived models and are strictly connected with nested and crossed structures of experimental units. Statistical properties result from the types of randomization schemes applied in the experiments.

The aim of the paper is comparing both experiments with respect to the relative efficiency in order to estimation and testing hypotheses for factors and interactions between them. It should be noted that selecting experimental design, we follow mainly possibility (or ease) to apply it in practice. Additionally an order of factors plays an important role in the experiment. However, economic considerations and practical reasons, cannot obscure statistical purposes of the designs. This subject was raised also in the papers of Ambroży and Mejza (2006, 2008a, 2008b, 2012b). The considerations are illustrated with an example involving a winter wheat trial.

2. Structure of an experimental material

Consider an $s \times t \times w$ experiment designed to test the effects of two (s = 2) levels of nitrogen fertilization (kg/ha) A_1 – 90, A_2 – 150 and two (w = 2) chemical growth regulators (kg/ha) C_1 – 0, C_2 – 2 on the grain yields of five (t = 5) wheat varieties B_1 – Grana, B_2 – Dana, B_3 – Eka Nowa, B_4 – Kaukaz, B_5 – Mironowskaja 808. So we have v = 20 treatment combinations. The original experiment was performed in Słupia Wielka (Poland) on a complete split-plot × split-block (say, SPSB) design, in three blocks (replications); see Mucha (1975). The structure of the experimental material (in this case: of the field) is described in 2.1. Then for comparison, the same data were analyzed under a mixed linear model as data from a complete split-split-plot (say, SSP) design (see 2.2). All analyzes were performed with the help of the STATISTICA package (Ambroży and Mejza, 2006, 2012b).

2.1. Split-plot × split-block (SPSB) design

In the SPSB design every block (b = 3) of the experimental material forms a row-column design with s (= 2) rows and t (= 5) columns of the first order, called I-columns for short. Then each I-column has to be split into w (= 2) columns of the second order (called II-columns). In this case, the rows also correspond to the levels of factor A, termed row treatments, the I-columns correspond to the levels of factor B, termed I-column treatments, and the II-columns are to accommodate the levels of factor C, termed II-column treatments. Here, the third factor is in a split-plot design in a relation to the I-column treatments (which in turn are in a split-block design with the row treatments).

2.2. Split-split-plot (SSP) design

In the SSP design it is assumed that the experimental material can be divided into b (= 3) blocks. Every block can be divided into s (= 2) whole plots. Then, each whole plot is divided into t (= 5) subplots and then each subplot can be divided into w (= 2) sub-subplots. Here the whole plots correspond to the levels of factor A (whole plot treatments), the subplots correspond to the levels of factor B (subplot treatments), and the sub-subplots are to accommodate the levels of factor C (sub-subplot treatments). Hence the third factor is in a split-plot design in relation to the whole plot and subplot treatment combinations (i.e. combinations of the levels of factor A and factor B which are also in a split-plot design).

In both SPSB and SSP designs every block consists of $stw = v = 20$ plots and the number of observations is equal to $n = bstw = 60$.

3. Mixed linear models

We consider a randomized-derived models of observations of which the forms and properties are strictly connected with performed randomization processes in the experiment. The randomization schemes used here consist of four randomization steps performed independently. Different schemes of randomization in the considered designs lead to the following models

$$\mathbf{y} = \Delta'\tau + \sum_{f=1}^{m} \mathbf{D}'_f \xi_f + \mathbf{e}, \tag{3.1}$$

where m = 6 for the SPSB design (cf. Ambroży and Mejza, 2003, 2004) and m = 4 for the SSP design (Mejza, 1997a, 1997b). The considered models of the form (3.1) have the following properties:

$$\mathrm{E}(\mathbf{y}) = \Delta'\boldsymbol{\tau}, \qquad \mathrm{Cov}(\mathbf{y}) = \sum_{f=1}^{m} \mathbf{D}'_f \mathbf{V}_f \mathbf{D}_f + \sigma_e^2 \mathbf{I}_n, \tag{3.2}$$

where Δ' is a known design matrix for v treatment combinations, and τ (v×1) is the vector of fixed treatment combination parameters. According to the orthogonal block structure of the considered designs, the covariance matrix Cov(y) can be expressed by

$$\mathrm{Cov}(\mathbf{y}) = \gamma_0\mathbf{P}_0 + \gamma_1\mathbf{P}_1 + \gamma_2\mathbf{P}_2 + \ldots + \gamma_m\mathbf{P}_m, \tag{3.3}$$

where $\gamma_f \geq 0$ and $\{\mathbf{P}_f\}$ is a family of known pairwise orthogonal matrices summing to the identity matrix (cf. Houtman and Speed, 1983). The range space of $\mathbf{P}_f$ for f = 0, 1,.., m, is termed the f-th stratum of the model, and $\{\gamma_f\}$ are unknown strata variances. They are some functions of unknown variance components resulting from the suitable for the design scheme of randomization.

In the SPSB they are:

$$\gamma_0 = \sigma_e^2, \quad \gamma_1 = stw\sigma_1^2 + \sigma_e^2, \quad \gamma_2 = tw\sigma_2^2 + \sigma_e^2,$$

$$\gamma_3 = sw\sigma_3^2 + \sigma_e^2, \quad \gamma_4 = s\sigma_4^2 + \sigma_e^2, \tag{3.4}$$

$$\gamma_5 = w\sigma_5^2 + \sigma_e^2, \quad \gamma_6 = \sigma_6^2 + \sigma_e^2,$$

where σ_f^2 (f = 1, 2,..., 6) denote variances components of effects of blocks, rows, columns I, columns II, whole plots and subplots, respectively and σ_e^2 means a variance component of a technical error.

In the SSP design they are:

$$\gamma_0 = \sigma_e^2, \quad \gamma_1 = stw\sigma_1^2 + \sigma_e^2,$$

$$\gamma_2 = tw\sigma_2^2 + \sigma_e^2, \quad \gamma_3 = w\sigma_3^2 + \sigma_e^2, \quad \gamma_4 = \sigma_4^2 + \sigma_e^2, \tag{3.5}$$

where σ_f^2 (f = 1, 2, 3, 4) denote variance components of effects of blocks, whole plots, subplots, sub-subplots, respectively and σ_e^2 means a variance component of a technical error.

Models (3.1) can be analyzed using the methods developed for multistratum experiments (see, e.g., Ambroży and Mejza, 2006).

4. Relative efficiency and empirical relative efficiency

Definition 4.1. Let Γ_1 and Γ_2 denote any experimental designs, then the relative efficiency of Γ_1 and Γ_2 is defined as (see Yates, 1935)

$$\mathrm{RE}(\Gamma_1 / \Gamma_2) = \frac{\textit{Efficiency}\ \Gamma_1}{\textit{Efficiency}\ \Gamma_2} = \frac{\mathrm{Var}\Gamma_2}{\mathrm{Var}\Gamma_1}, \tag{4.1}$$

where $\mathrm{Var}\Gamma_1$ and $\mathrm{Var}\Gamma_2$ denote the variances of the same contrast in the respective designs.

The relative efficiency as defined in (4.1) depends on the true stratum variances $\{\gamma_f\}$ of the designs, which are usually unknown. Moreover, the stratum variances are functions of variance components defined during the randomization processes. Relations among them allow comparison of the efficiencies of the considered designs. Usually the same relations occur among estimates of the stratum variances (**except for sampling errors**). Comparing efficiency of the SSP design in relation to efficiency of the SPSB design for an estimation of contrasts, in some cases we can use the measure defined in (4.1), but in others we should take into account the estimation of the RE, called empirical relative efficiency, which we shall denote by ERE (see (4.2)).

Following Yates (1935) we consider uniformity trials, that is, trials with dummy treatments, with b blocks and experimental units in each block (see also, Hinkelmann and Kempthorne, 2008). The ANOVA tables for data with structures corresponding to SPSB design and SSP design, are given in Table 1 and Table 2, respectively.

Definition 4.2. Empirical relative efficiency (ERE) is defined as follows: (cf. Shieh and Jan, 2004; Wang and Hering, 2005)

$$\mathrm{ERE}_h(SSP/SPSB) = \frac{\widehat{\mathrm{Var}}^{SPSB}[(\mathbf{c}'_h\hat{\boldsymbol{\tau}})_f]}{\widehat{\mathrm{Var}}^{SSP}[(\mathbf{c}'_h\hat{\boldsymbol{\tau}})_{f'}]} = \frac{\hat{\gamma}_f^{SPSB}}{\hat{\gamma}_{f'}^{SSP}}, \tag{4.2}$$

where $\hat{\gamma}_f$ - estimates of variance components in the SSP and SPSB designs (see Tables 1–2), $h \in K = K_A \cup K_B \cup K_C \cup K_{A\times B} \cup K_{A\times C} \cup K_{B\times C} \cup K_{A\times B\times C}$, where $K = \{h : h = 1, 2, ..., v-1\}$ and $K_A, K_B, K_C, K_{A\times B}, K_{A\times C}, K_{B\times C}, K_{A\times B\times C}$

denote sets of numbers of orthogonal contrasts connected with main effects and different types of interaction effects of the factors A, B, C and $f = 1, 2, ..., m$, $f' = 1, 2, ..., m'$.

Table 1. ANOVA for dummy experiment in the SPSB design

Source	D.f.	*Mean Square*
(1) Blocks	$b - 1$	$\hat{\gamma}_1 = MSE_1$
(2) Rows	$b(s - 1)$	$\hat{\gamma}_2 = MSE_2$
(3) I-columns	$b(t - 1)$	$\hat{\gamma}_3 = MSE_3$
(4) II-columns	$bt(w - 1)$	$\hat{\gamma}_4 = MSE_4$
(5) Whole plots	$b(s - 1)(t - 1)$	$\hat{\gamma}_5 = MSE_5$
(6) Subplots	$bt(s - 1)(w - 1)$	$\hat{\gamma}_6 = MSE_6$
Total	$n - 1$	

Table 2. ANOVA for dummy experiment in the SSP design

Source	D.f.	*Mean Square*
(1) Blocks	$b - 1$	$\hat{\gamma}_1 = MSE_1$
(2) Whole plots	$b(s - 1)$	$\hat{\gamma}_2 = MSE_2$
(3) Subplots	$bs(t - 1)$	$\hat{\gamma}_3 = MSE_3$
(4) Sub-subplots	$bst(w - 1)$	$\hat{\gamma}_4 = MSE_4$
Total	$n - 1$	

Following Ambroży and Mejza (2006, 2008b) and from (3.4) - (3.5) it can be assumed (**except for sampling errors**) that the estimates of variance components $\hat{\gamma}_f$ in the designs satisfy the following inequalities:

for SPSB design: $\hat{\gamma}_1 > \hat{\gamma}_2 > \hat{\gamma}_5 > \hat{\gamma}_6$ and $\hat{\gamma}_1 > \hat{\gamma}_3 > \hat{\gamma}_4 > \hat{\gamma}_5 > \hat{\gamma}_6$,

for SSP design: $\hat{\gamma}_1 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_4$. (4.3)

Additionally, it can be expected that the appropriate stratum variances, and hence also their estimates, in the designs will be identical (to some extent), i.e.

$$\hat{\gamma}_1^{\mathrm{SPSB}} = \hat{\gamma}_1^{\mathrm{SSP}}, \qquad \hat{\gamma}_2^{\mathrm{SPSB}} = \hat{\gamma}_2^{\mathrm{SSP}} \tag{4.4}$$

It should be pointed out that $\hat{\gamma}_1^{\mathrm{SPSB}} = \hat{\gamma}_1^{\mathrm{SSP}}$, if the designs are complete (as for instance in this paper). Then, it can be shown that the subplot error and the sub-subplot error in the SSP design is a weighted average between the two errors in the SPSB design, i.e.

$$\hat{\gamma}_5^{\mathrm{SPSB}} < \hat{\gamma}_3^{\mathrm{SSP}} < \hat{\gamma}_3^{\mathrm{SPSB}}, \qquad \hat{\gamma}_6^{\mathrm{SPSB}} < \hat{\gamma}_4^{\mathrm{SSP}} < \hat{\gamma}_4^{\mathrm{SPSB}}. \tag{4.5}$$

5. Analysis using the STATISTICA software package

In Tables 3 and 4 given below we present STATISTICA output for a mixed model analyses of the grain yields of wheat. The effects of blocks are defined as random effects and the remaining effects are fixed (except of the errors).

Table 3. ANOVA for the complete SPSB design

Source	Effect	SS	DF	MS	F	p
Blocks	Randon	165.952	2	82.976		
A	Fixec	302.850	1	302.850	102.45	0.0096
Error (2)	Randon	5.912	2	2.956		
B	Fixec	493.612	4	123.403	47.09	0.000
Error (3)	Randon	20.962	8	2.620		
C	Fixec	216.600	1	216.600	174.10	0.000
B x C	Fixec	58.171	4	14.542	11.69	0.000
Error (4)	Randon	12.018	10	1.201		
A x B	Fixec	28.751	4	7.187	3.13	0.079
Error (5)	Randon	18.366	8	2.295		
A x C	Fixec	2.090	1	2.090	2.12	0.176
A x B x C	Fixec	13.674	4	3.418	3.46	0.050
Error (6)		9.875	10	0.987		

Table 4. ANOVA for the complete SSP design

Source	Effect	*SS*	*DF*	*MS*	*F*	*p*
Blocks	Randon	165.952	2	82.976		
A	Fixec	302.850	1	302.850	102.4	0.009
Error (2)	Randon	5.912	2	2.956		
B	Fixec	493.612	4	123.403	50.2	0.000
A x *B*	Fixec	28.751	4	7.187	2.92	0.054
Error (3)	Randon	39.328	16	2.458		
C	Fixec	216.600	1	216.600	197.8	0.000
A x *C*	Fixec	2.090	1	2.090	1.91	0.182
B x *C*	Fixec	58.171	4	14.542	13.2	0.000
A x *B* x *C*	Fixec	13.674	4	3.418	3.12	0.037
Error (4)		21.893	20	1.094		

6. Results and conclusions

The relations (4.3)–(4.5) were applied to examine the empirical relative efficiency of the complete SSP and SPSB designs for an estimation of the orthogonal contrasts associated with the main and interaction effects of the factors.

1) For estimation of the contrasts associated with the main effects of factor *A* (Nitrogen fertilization) both considered designs are equally effective

$$ERE_h^A(SSP/SPSB) = \frac{\widehat{\mathrm{Var}}^{SPSB}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_2]}{\widehat{\mathrm{Var}}^{SSP}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_2]} = \frac{\hat{\gamma}_2^{SPSB}}{\hat{\gamma}_2^{SSP}} = 1, \text{ where } h \in K_A.$$

2) The SSP design is more effective than the SPSB design for estimation of the contrasts associated with the main effects of factor *B*

$$ERE_h^B(SSP/SPSB) = \frac{\widehat{\mathrm{Var}}^{SPSB}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_3]}{\widehat{\mathrm{Var}}^{SSP}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_3]} = \frac{\hat{\gamma}_3^{SPSB}}{\hat{\gamma}_3^{SSP}} > 1, \text{ where } h \in K_B.$$

3) The SSP design is less effective than SPSB design for estimation of the interaction contrasts of type $A \times B$

$$ERE_h^{A\times B}(SSP/SPSB) = \frac{\widehat{\mathrm{Var}}^{SPSB}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_f]}{\widehat{\mathrm{Var}}^{SSP}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_3]} = \frac{\hat{\gamma}_f^{SPSB}}{\hat{\gamma}_3^{SSP}} < 1, \text{ where } h \in K_{A\times B}, f = 4, 5.$$

4) The SPSB design is more effective than the SSP design in estimation of the interaction contrasts of types $A \times C$ and $A \times B \times C$

$$ERE_h(SSP/SPSB) = \frac{\hat{\mathrm{Var}}^{SPSB}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_6]}{\hat{\mathrm{Var}}^{SSP}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_4]} = \frac{\hat{\gamma}_6^{SPSB}}{\hat{\gamma}_4^{SSP}} < 1, \text{ where } h \in K_{A\times C} \cup K_{A\times B\times C}.$$

5) The SSP design is more effective than the SPSB design for estimation of the contrasts associated with the main effects of factor C and the interaction effects of type $B \times C$

$$ERE_h(SSP/SPSB) = \frac{\hat{\mathrm{Var}}^{SPSB}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_4]}{\hat{\mathrm{Var}}^{SSP}[(\boldsymbol{c}_h'\hat{\boldsymbol{\tau}})_4]} = \frac{\hat{\gamma}_4^{SPSB}}{\hat{\gamma}_4^{SSP}} > 1, \text{ where } h \in K_C \cup K_{B\times C}.$$

A summary of the conclusions given above is presented in Table 5. The symbols *a*, *b* are used to denote efficiency levels in descending order.

Table 5. Comparing of the efficiency of the SSP design and SPSB design in the estimation of some groups of contrasts

Sources	SSP	SPSB
A	***a***	***a***
B	***a***	*b*
$A \times B$	*b*	***a***
C	***a***	*b*
$A \times C$	*b*	***a***
$B \times C$	***a***	*b*
$A \times B \times C$	*b*	***a***

References

Ambroży K., Mejza I. (2003). Some split-plot × split-block designs. *Colloquium Biometryczne* 33, 83–96.

Ambroży K., Mejza I. (2004). Split-plot × split-block type three factors designs. Proc. of the 19th International Workshop on Statistical Modelling, Florence, 291–295.

Ambroży K., Mejza I. (2006). *Doświadczenia trójczynnikowe z krzyżową i zagnieżdżoną strukturą poziomów czynników.* Wyd. PTB i PRODRUK, Poznań.

Ambroży K., Mejza I. (2008a). Mixed designs in agriculture. XVIIIth Summer School of Biometrics, Pavel Fl'ak (Ed.),VUZV, Nitra, Slovakia, 27–35.

Ambroży K., Mejza I. (2008b). The relative efficiency of split–plot × split–block designs and split–block–plot designs. *Biometrical Letters* 45, No. 1, 29–44.

Ambroży K., Mejza I. (2009). Analiza danych z krzyżową i zagnieżdżoną strukturą poziomów na przykładzie doświadczenia z łubinem. *Biul. IHAR*, Nr 251, 269–281.

Ambroży K., Mejza I. (2012a). On the efficiency of some non-orthogonal split-plot × split-block designs with control treatments. *Journal of Statistical Planning and Inference*. Vol. 142, Issue 3, 752–762.

Ambroży K., Mejza I. (2012b). Modelowanie danych z doświadczeń trójczynnikowych zakładanych w układach zależnych o różnych strukturach blokowych. *Biul. IHAR* 264, 23–31.

Hinkelmann K., Kempthorne O. (2008). *Design and Analysis of Experiments*. Wiley, New Jersey.

Houtman A.M., Speed T.P. (1983). Balance in designed experiments with orthogonal block structure. *Ann. Statist*. 11, 1069–1085.

Mejza I. (1997*a*). Doświadczenia trójczynnikowe w niekompletnych układach o poletkach podwójnie rozszczepionych. *Colloquium Biometryczne* 27, 127–138.

Mejza I. (1997*b*). Incomplete split-split-plot designs. 51st Session of the International Statistical Institute ISI'97, Istanbul. Contributed papers, book 1, 221–222.

Mucha S. (1975). Reakcja odmian pszenicy jarej i ozimej na antywylęgacz. *Wiadomości odmianoznawcze*, rok II, zeszyt 2/3, COBORU, Słupia Wielka.

Shieh G., Jan S.L. (2004). The effectiveness of randomized complete block design. *Statistica Neerlandica* 58, No. 1, 111–124.

Wang M., Hering F. (2005). Efficiency of split-block designs versus split-plot designs for hypothesis testing. *Journal of Statistical Planning and Inference* 132, 163–182.

Yates F. (1935). Complex experiments (with discussion). *J. Roy. Statist. Soc*. Suppl. 2, 181–247.

StatSoft, Inc. (2011). STATISTICA (data analysis software system), version 10. www.statsoft.com.