Colloquium Biometricum 45
2015, 23–34

RELATIONS BETWEEN TERNARY DESIGNS AND D-OPTIMAL WEIGHING DESIGNS WITH NON-NEGATIVE CORRELATED ERRORS

Bronisław Ceranka, Małgorzata Graczyk

Department of Mathematical and Statistical Methods
Poznań University of Life Sciences
Wojska Polskiego 28, 60-637 Poznań, Poland
e-mails: bronicer@up.poznan.pl, magra@up.poznan.pl

Summary

This work was intended as an attempt at motivating to increase the list of possible classes in that we are able to construct optimal plans of experiments. Here, we consider the criterion of D-optimality. We introduce the relations between ternary balanced block designs and chemical balance weighing designs. Based on these relations we propose to construct the design matrix of the regular D-optimal chemical balance weighing design on the base of the set of the incidence matrices of the ternary balanced block designs. We consider these designs under basic assumption that the errors are equally correlated and they have the same variances.

Key words and phrases: chemical balance weighing design, D-optimality, ternary balanced block design

Classification AMS 2010: 62K05, 62K10

1. Introduction

The chemical balance weighing design is described by the model $\mathbf{y} = \mathbf{Xw} + \mathbf{e}$, where $\mathbf{y}$ is an $n \times 1$ random vector of the observations,

$\mathbf{w} = (w_1, w_2, ..., w_p)'$ is a vector representing unknown measurements of p objects, $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$, where $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ denotes the class of $n \times p$ matrices having entries $x_{ij} = -1, 1$ or 0, m $(\leq n)$ is the maximal number of elements equal to 1 and –1 in each column of the matrix $\mathbf{X}$. It is worth emphasising, that the class $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ is explicitly defined throughout the parameters n and p. The parameter m is subsidiary and in extreme cases, $m = n$. We shall make two standing assumptions on the maps under considerations. The errors of measurements are equally correlated and they have the same variances. These two assumptions we can describe in the form: for $n \times 1$ random vector of errors $\mathbf{e}$, $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$, $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where

$$\mathbf{G} = g\left((1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n'\right), \; g > 0, \; 0 \leq \rho < 1. \tag{1.1}$$

Note, for $0 \leq \rho < 1$ and $g > 0$ the matrix $\mathbf{G}$ is positive definite. We can assume that the matrix $\mathbf{X}$ is of full column rank. In this situation all w_j, $j = 1,2,..., p$, are estimable and the variance matrix of the best linear unbiased estimator $\hat{\mathbf{w}} = \mathbf{M}^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ is equal $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\mathbf{M}^{-1}$, where $\mathbf{M} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix for the design $\mathbf{X}$.

In the theory of weighing designs, there appear manifold problems. In Jacroux and Notz (1983), Jacroux et al. (1983), Neubauer and Pace (2010), Graczyk (2012) and Ceranka and Graczyk (2014a), the issues concerned on the setting down the conditions determining optimal design for different optimality criteria are presented. The questions considered in Sathe and Shenoy (1990), Ceranka and Graczyk (2002) and Li and Yang (2005) lead our attention to the construction methods optimal designs. The matter of application of weighing designs is dealt in Koukouvinos and Seberry (1997), Banerjee and Mukerjee (2007) and Graczyk (2014).

The aim of present research is to plan the experiment in order to attain the best estimators of unknown measurements of objects. For that reason, the optimal designs are determined. They permit to estimate unknown parameters with minimal variance. Thus, different optimality criteria minimize some functions of the matrix $\mathbf{M}$. Here, we develop the theory of D-optimal designs. The D-optimality criterion is also known as determinant criterion and results in minimizing the generalized variance of the parameter estimates, see Raghavarao (1971), Shah and Sinha (1989), Masaro and Wong (2008A), Katulska and Smaga (2013) and Smaga (2014). Let note, if the observation vector follows a multivariate normal distribution, then D-optimality criterion chooses that design as the “best” for which the volume (expected volume) of the joint confidence ellipsoid is least.

In the class $\mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$, the design $\mathbf{X}_D$ is D-optimal if $\det\left(\mathbf{X}_D'\mathbf{G}^{-1}\mathbf{X}_D\right)=\min\left(\det\left(\mathbf{M}^{-1}\right):\mathbf{X}\in\mathbf{\Phi}_{n\times p,m}\{-1,0,1\}\right)$. If $\det\left(\mathbf{X}_D'\mathbf{G}^{-1}\mathbf{X}_D\right)$ attains the lowest bound then the design is called regular D-optimal. In other cases it is called D-optimal. It is worth noting, regular D-optimal design is also D-optimal, but the converse isn’t necessarily true.

Under assumption that the matrix $\mathbf{G}$ in $\sigma^2\mathbf{G}$ is of the form (1.1), in Ceranka and Graczyk (2014a), the conditions determining regular D-optimal chemical balance weighing design and the construction methods based on the incidence matrices of the balanced incomplete block designs are collected. Consequently, we mention the definition and the theorem concerned the D-optimal design given in above paper for this particular case.

Definition 1.1. Any chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1), is regular D-optimal if $\det\left(\mathbf{M}^{-1}\right)=\left(\frac{g(1-\rho)}{m}\right)^p$.

Theorem 1.1. Any chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1), is regular D-optimal if and only if

(i) $\mathbf{X}'\mathbf{X}=m\mathbf{I}_p$ if $\rho=0$,

(ii) $\mathbf{X}'\mathbf{X}=m\mathbf{I}_p$ and $\mathbf{X}'\mathbf{1}_n=\mathbf{0}_p$ if $0<\rho<1$.

The conditions in (ii) mean that m has to be even and in each column of $\mathbf{X}$ the numbers of elements –1 should be the same as +1.

In present considerations, we investigate the methods of determining the classes $\mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$ in those the regular D-optimal design exists. Subsequently, we look more closely at the problems concerned on the construction methods of D-optimal designs. We are focus on indicating some known designs based on their incidence matrices we are able to construct the design matrix of regular D-optimal design.

2. The construction

Masaro and Wong (2008A) and Katulska and Smaga (2013) derived the results of construction of regular D-optimal weighing designs in the class $\mathbf{\Pi}_{n\times p}\{-1, 1\}$, where $\mathbf{\Pi}_{n\times p}\{-1, 1\}$ denotes the class of $n\times p$ matrices having entries $x_{ij} = -1$ or 1. Ceranka and Graczyk (2014b) extend these results to the class $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$. They give the construction method of regular D-optimal weighing design based on the incidence matrices of the balanced bipartite weighing designs. Unfortunately, it is not possible to construct the design matrix of regular D-optimal weighing design in any class $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$. In some classes $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$, we cannot determine optimal design based on the methods given in Ceranka and Graczyk (2014b). For that reason, in this section we give new construction based on the set of the incidence matrices of ternary balanced block designs (see Billington, 1984). The advantage of using ternary balanced block designs lies in the fact that based on their incidence matrices we are able to construct the matrix $\mathbf{X}\in\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ satisfying optimality criterion. Now, we recall the basic facts pertain to these designs.

Any ternary balanced block design there is an arrangement of v treatments in b blocks, each of size k in such a way that each treatment appears 0, 1 or 2 times in r blocks. Each of the distinct pairs of treatments appears λ times. Any ternary balanced block design is regular, that is, each treatment occurs alone in ρ_1 blocks and is repeated two times in ρ_2 blocks, where ρ_1 and ρ_2 are constant for the design. It is straightforward to verify that $vr = bk$, $r = \rho_1 + 2\rho_2$, $\lambda(v-1) = \rho_1(k-1) + 2\rho_2(k-2)$. The incidence matrix $\mathbf{N}$ of such a design has elements equal to 0, 1 or 2 and moreover, $\mathbf{NN}' = (\rho_1 + 4\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$.

Now, in order to determine regular D-optimal weighing design in new classes $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$, let us construct the design matrix $\mathbf{X}$ of the chemical balance weighing design from the set of t incidence matrices of the ternary balanced block designs with parameters v, b_h, r_h, k_h, λ_h, ρ_{1h}, ρ_{2h}, $h = 1,2,...,t$. So, we obtain

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'} \\ \mathbf{N}_2^{'} - \mathbf{1}_{b_2}\mathbf{1}_v^{'} \\ \cdots \\ \mathbf{N}_t^{'} - \mathbf{1}_{b_t}\mathbf{1}_v^{'} \end{bmatrix}. \tag{2.1}$$

In this matrix, each column contains $\sum_{h=1}^{t} \rho_{2h}$ elements equal to 1, $\sum_{h=1}^{t} (b_h - \rho_{1h} - \rho_{2h})$ elements equal to –1 and $\sum_{h=1}^{t} \rho_{1h}$ elements equal to 0. Clearly, such form of the design implies that each object is weighted $m = \sum_{h=1}^{t} (b_h - \rho_{1h})$ times in the $n = \sum_{h=1}^{t} b_h$ weighing operations. Hence, from practical point of view, the number t should not be extremely large.

Lemma 2.1. Any chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) is nonsingular if and only if $v \neq k_h$ for at least one $h = 1,2,...,t$.

Proof. Any chemical balance weighing design is called nonsingular if its design matrix is of full column rank. In this situation, all unknown measurements of objects are estimable. If the matrix $\mathbf{G}$ is positive definite, then the matrix $\mathbf{M}$ is nonsingular if and only if $\mathbf{X}^{'}\mathbf{X}$ is nonsingular. For $\mathbf{G}$ of the form (1.1) and $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given in (2.1), we have

$$\mathbf{X}^{'}\mathbf{X} = \left[\sum_{h=1}^{t} (r_h - \lambda_h + 2\rho_{2h})\right]\mathbf{I}_v + \left[\sum_{h=1}^{t} (b_h + \lambda_h - 2r_h)\right]\mathbf{1}_v\mathbf{1}_v^{'}. \tag{2.2}$$

Moreover, we obtain

$$\det(\mathbf{X}^{'}\mathbf{X}) = \left[\sum_{h=1}^{t} (r_h - \lambda_h + 2\rho_{2h})\right]^{v-1} \left[\sum_{h=1}^{t} \frac{r_h}{k_h}(v - k_h)^2\right]. \tag{2.3}$$

Because $\sum_{h=1}^{t}(r_h - \lambda_h + 2\rho_{2h}) > 0$, then the determinant (2.3) equals 0 if and only if $v = k_h$ for each $h = 1,2,...,t$. Thus the lemma is proved. ■

Theorem 2.1. Let $\rho = 0$. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$, is regular D-optimal if and only if

$$\sum_{h=1}^{t}(b_h + \lambda_h - 2\rho_{2h}) = 0 . \tag{2.4}$$

Proof. We have been working under assumption $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ is given by (2.1) and $\mathbf{G} = \mathbf{I}_n$. Thus we obtain that the product $\mathbf{X}'\mathbf{X}$ is of the form (2.2). On the base of Theorem 1.1(i), $\mathbf{X}$ is the design matrix of the regular D-optimal chemical balance weighing design if and only if $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$. From the above it follows that $\sum_{h=1}^{t}(b_h + \lambda_h - 2\rho_{2h}) = 0$ which is the desired conclusion. ■

Now, we can assume that $b_h = 2r_h - \lambda_h$ for each $h = 1,2,...,t$. Therefore, we obtain the following corollary.

Corollary 2.1. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$ is regular D-optimal if and only if $b_h = 2r_h - \lambda_h$ for each $h = 1,2,...,t$.

We place emphasis on construction the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ of the regular D-optimal chemical balance weighing design. There is a big number of receivable combinations between the parameters of the ternary balanced block designs for that the condition $b_h = 2r_h - \lambda_h$ holds. For $t = 1$, some method of construction $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) for which the condition $b = 2r - \lambda$ holds, was given by Ceranka et al. (1998). Now, we generalize this result.

Theorem 2.2. Let $\rho = 0$. If the ternary balanced block design with the parameters

(i) $v = s$, $b = us$, $r = u(s-2)$, $k = s-2$, $\lambda = u(s-4)$, $\rho_1 = u(s-4)$, $\rho_2 = u$, $s = 5,6,...$, $u = 1,2,...$, except the case $u = 1$ and $s = 5$,

(ii) $v = s$, $b = us$, $r = u(s-3)$, $k = s-3$, $\lambda = u(s-6)$, $\rho_1 = u(s-9)$, $\rho_2 = 3u$, $s = 10,11,...$, $u = 1,2,...$,

(iii) $v = s$, $b = us$, $r = u(s-4)$, $k = s-4$, $\lambda = u(s-8)$, $\rho_1 = u(s-16)$, $\rho_2 = 6u$, $s = 17,18,...$, $u = 1,2,...$,

(iv) $v = 12s$, $b = 4us$, $r = u(4s-1)$, $k = 3(4s-1)$, $\lambda = 2u(2s-1)$, $\rho_1 = u(4s-3)$, $\rho_2 = u$, $s = 1,2,...$, $u = 4,5,...$,

exists, then $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) is the regular D-optimal chemical balance weighing design with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$.

Proof. It is easy to verify that the parameters of ternary balanced block designs given in (i)-(iv) satisfy the condition $b = 2r - \lambda$. ■

Next, we consider the case $t = 2$. Here, optimality condition given in Theorem 2.1 has the form

$$b_1 + b_2 = 2r_1 + 2r_2 - \lambda_1 - \lambda_2 . \tag{2.5}$$

We formulate theorem presenting series of the parameters of two ternary balanced block designs that satisfy the condition (2.5). Based on these parameters we form the incidence matrices $\mathbf{N}_h$, $h = 1,2$, of the ternary balanced block designs and afterwards the matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ of the regular D-optimal chemical balance weighing design with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$.

Theorem 2.3. Let $\rho = 0$. If the ternary balanced block designs with the parameters

(i) $v = 5$, $b_1 = r_1 = u + 4$, $k_1 = 5$, $\lambda_1 = u + 3$, $\rho_{11} = u$, $\rho_{21} = 2$ and

$v=5$, $b_2=5(s+2)$, $r_2=3(s+2)$, $k_2=3$, $\lambda_2=s+3$, $\rho_{12}=s+6$, $\rho_{22}=s$, $s,u=1,2,\ldots$,

(ii) $v=5$, $b_1=r_1=u+9$, $k_1=5$, $\lambda_1=u+7$, $\rho_{11}=u+1$, $\rho_{21}=4$ and $v=5$, $b_2=5(s+4)$, $r_2=3(s+4)$, $k_2=3$, $\lambda_2=s+6$, $\rho_{12}=s+12$, $\rho_{22}=s$, $s,u=1,2,\ldots$,

(iii) $v=5$, $b_1=5(s+1)$, $r_1=4(s+1)$, $k_1=4$, $\lambda_1=3s+2$, $\rho_{11}=4s$, $\rho_{21}=2$ and $v=5$, $b_2=5(u+2)$, $r_2=3(u+2)$, $k_2=3$, $\lambda_2=u+3$, $\rho_{12}=u+6$, $\rho_{22}=u$, $s,u=1,2,\ldots$,

(iv) $v=6$, $b_1=r_1=u+10$, $k_1=6$, $\lambda_1=u+8$, $\rho_{11}=u$, $\rho_{21}=5$ and $v=6$, $b_2=2(s+5)$, $r_2=s+5$, $k_2=3$, $\lambda_2=2$, $\rho_{12}=5-s$, $\rho_{22}=s$, $s=1,2,3,4$, $u=1,2,\ldots$,

(v) $v=7$, $b_1=r_1=u+13$, $k_1=7$, $\lambda_1=u+11$, $\rho_{11}=u+1$, $\rho_{21}=6$ and $v=7$, $b_2=21$, $r_2=12$, $k_2=4$, $\lambda_2=5$, $\rho_{12}=6$, $\rho_{22}=3$, $u=1,2,\ldots$,

(vi) $v=9$, $b_1=r_1=u+8$, $k_1=9$, $\lambda_1=u+7$, $\rho_{11}=u$, $\rho_{21}=4$ and $v=9$, $b_2=3(s+4)$, $r_2=2(s+4)$, $k_2=6$, $\lambda_2=s+5$, $\rho_{12}=8$, $\rho_{22}=s$, $s,u=1,2,\ldots$,

(vii) $v=k_1=11$, $b_1=r_1=u+10$, $\lambda_1=u+9$, $\rho_{11}=u$, $\rho_{21}=5$ and $v=b_2=11$, $r_2=k_2=7$, $\lambda_2=4$, $\rho_{12}=5$, $\rho_{22}=1$, $u=1,2,\ldots$,

(viii) $v=12$, $b_1=18$, $r_1=15$, $k_1=10$, $\lambda_1=11$, $\rho_{11}=1$, $\rho_{21}=7$ and $v=12$, $b_2=3(2s+5)$, $r_2=2(2s+5)$, $k_2=8$, $\lambda_2=2(s+3)$, $\rho_{12}=6-2s$, $\rho_{22}=3s+2$, $s=0,1,2$,

(ix) $v=k_1=15$, $b_1=r_1=u+14$, $\lambda_1=u+13$, $\rho_{11}=u$, $\rho_{21}=7$ and $v=15$, $b_2=3(s+4)$, $r_2=2(s+4)$, $k_2=10$, $\lambda_2=s+5$, $\rho_{12}=6-2s$, $\rho_{22}=2s+1$, $s=1,2$, $u=1,2,\ldots$,

exist, then $\mathbf{X}\in\mathbf{\Phi}_{n\times p,m}\{-1,0,1\}$ given by (2.1) is the regular D-optimal chemical balance weighing design with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$.

Proof. The parameters of the ternary balanced block designs (i)-(ix) satisfy the condition (2.5). ■

In particular case, when $\mathbf{N}_1$ is the incidence matrix of the ternary balanced block design with the parameters v, b_1, r_1, k_1, λ_1, ρ_{11}, ρ_{21} and $\mathbf{N}_2$ is the incidence matrix of its complementary design, i.e. $\mathbf{N}_2 = 2\mathbf{1}_v\mathbf{1}_{b_1}^{'} - \mathbf{N}_1$ with the parameters v, $b_2 = b_1$, $r_2 = 2b_1 - r_1$, $k_2 = 2v - k_1$, $\lambda_2 = \lambda_1 + 4b_1 - 4r_1$, $\rho_{12} = \rho_{11}$, $\rho_{22} = b_1 - \rho_{11} - \rho_{21}$, then the condition (2.5) reduces to $b_1 = 2r_1 - \lambda_1$ and we have corollary.

Corollary 2.2. Let $\rho = 0$. If the ternary balanced block design with the parameters given in Theorem 2.2 exists, then $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ of the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'} \\ \mathbf{1}_{b_1}\mathbf{1}_v^{'} - \mathbf{N}_1^{'} \end{bmatrix} \tag{2.6}$$

is the regular D-optimal chemical balance weighing design with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$.

Now, we consider the case $0 < \rho < 1$. The conditions given in Theorem 1.1(ii) imply the following corollary.

Corollary 2.3. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$ for $0 < \rho < 1$, is regular D-optimal chemical balance weighing design if and only if $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2 g\mathbf{I}_n$ is regular D-optimal chemical balance weighing design under condition $\mathbf{X}^{'}\mathbf{1}_n = \mathbf{0}_p$.

From the construction of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ it follows that each column contains $\sum_{h=1}^{t} \rho_{2h}$ elements equal to 1 and $\sum_{h=1}^{t} (b_h - \rho_{1h} - \rho_{2h})$ elements equal to –1. Thus we have the following theorem.

Theorem 2.4. Let $0 < \rho < 1$. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ given by (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$ is regular D-optimal chemical balance weighing design if and only if the condition (2.4) is fulfilled and

$$\sum_{h=1}^{t} b_h = \sum_{h=1}^{t} r_h \,. \tag{2.7}$$

Proof. Assume the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ is of the form (2.1) and $\rho \in (0, 1)$. We conclude from Theorem 1.1(ii) that $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ and $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$. According to Theorem 2.1 the condition $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ is fulfilled if and only if (2.4) holds. Since $0 < \rho < 1$, the condition $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$ is true if and only in each column of $\mathbf{X}$, the number of elements equal to 1 is the same as the number of elements equal –1. In this way, we obtain (2.7). So, the Theorem is proved. ■

Corollary 2.4. Let $0 < \rho < 1$. If the ternary balanced block designs with the parameters given in Theorem 2.2 exist, then $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ of the form (2.6) with the covariance matrix of errors $\sigma^2\mathbf{G}$ for $0 < \rho < 1$, is regular D-optimal chemical balance weighing design.

3. Discussion

One of the basic problems in determining regular D-optimal chemical balance weighing designs in the given class $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$, is recommendation the construction method of such designs. The indicating new construction method of the matrix of optimal design is equivalent to the determining optimal plan of the experiment. In the literature, some incidence matrices of know block designs are used for constructions. For example, in Jacroux et al. (1983), the construction of $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ of the chemical balance weighing designs based on the incidence matrices of the balanced incomplete block designs, is presented. Under assumption that the errors are uncorrelated, i.e. $\rho = 0$, Ceranka and Graczyk (2014b) considered the construction of the design matrix $\mathbf{X}$ of the regular D-optimal chemical balance weighing design based on incidence matrices of balanced bipartite weighing designs. Based on this method, we are able to construct regular D-optimal chemical balance weighing design in the class $\mathbf{\Phi}_{36\times 9,16}\{-1, 0, 1\}$, i.e for $p = 9$ objects and $n = 36$ measurements (at least), see Theorem 2.3 in Ceranka and Graczyk (2014b) for $v = 9, u = 3$ and $s = 2$. Based on the method given in present paper, it is possible to determine unknown measurements of $p = 9$ objects in the regular D-optimal chemical balance weighing design for smaller number of measurements $n = 9$, i.e. in $\mathbf{\Phi}_{9\times 9,4}\{-1, 0, 1\}$, see Theorem 2.2(ii) for $s = 9$ and $u = 1$. Similarly, if $0 < \rho < 1$ and $p = 11$, in Ceranka and Graczyk (2014b) the regular D-optimal chemical balance weighing design exists in the class $\mathbf{\Phi}_{110\times 11,40}\{-1, 0, 1\}$, see Theorem 2.8 for $t = 11, u = 3$ and $s = 2$. Taking into consideration the results given above, we indicate regular D-optimal design in the class $\mathbf{\Phi}_{22\times 11,18}\{-1, 0, 1\}$, see Corollary 2.4 and Theorem 2.2(ii) for $s = 11$ and $u = 1$. Therefore, it seems to be advantageous, to indicate new classes in which the construction of optimal design is possible. We can strongly reduce the number of measurements and in consequence lessen experimental costs.

In the light of Corollary 2.4, if the ternary balanced block design with the parameters given in Theorem 2.2 exists, then the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ of the form (2.6) with the covariance matrix of errors $\sigma^2\mathbf{G}$ is optimal for any $0 < \rho < 1$. Thus, the design of such form is called robust, see Masaro and Wong (2008b). It means we can determine unknown measurements of objects according to the same D-optimal design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ for any $0 < \rho < 1$, i.e. for different correlation, it implies in various conditions.

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