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NEW CONSTRUCTION OF D-OPTIMAL WEIGHING DESIGNS WITH NON-NEGATIVE CORRELATIONS OF ERRORS

Bronisław Ceranka, Małgorzata Graczyk

Department of Mathematical and Statistical Methods
Poznań University of Life Sciences
Wojska Polskiego 28, 60-637 Poznań, Poland
emails: bronicer@up.poznan.pl, magra@up.poznan.pl

Summary

In this paper, the regular D-optimal chemical balance weighing designs with equally non-negative correlated errors are considered. Here we study the issues regard to the existence conditions of optimal designs. Presented construction method is based on the set of the incidence matrices of the balanced bipartite weighing designs and the ternary balanced block designs.

Key words and phrases: balanced bipartite weighing design, chemical balance weighing design, D-optimality, ternary balanced block design

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1. Introduction

Let us consider the linear model $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$, where $\mathbf{y}$ is a $n \times 1$ random vector of the observations, $\mathbf{w} = (w_1, w_2, ..., w_p)'$ is vector representing unknown measurements of p objects. The model is determined by the design matrix $\mathbf{X}$

and $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$, $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ denotes the class of $n \times p$ matrices having entries $-1, 1$ or 0, m $(\leq n)$ is the maximal number of elements equal 1 and –1 in each column of the matrix $\mathbf{X}$. The model of the chemical balance weighing design is equated to the matrix $\mathbf{X}$. Moreover, $\mathbf{e}$ is a $n \times 1$ random vector of errors and $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$, $\mathrm{Var}(\mathbf{e}) = \sigma^2 \mathbf{G}$,

$$\mathbf{G} = g\left((1-\rho)\mathbf{I}_n + \rho \mathbf{1}_n \mathbf{1}_n^{'}\right), \quad g > 0, \quad 0 \leq \rho < 1 \tag{1.1}$$

is positive definite matrix. Such form of the variance matrix of errors means that the errors of measurements are uncorrelated $(\rho = 0)$ or equally correlated and they have the same variances. If the matrix $\mathbf{X}$ is of full column rank then the estimator of the vector $\mathbf{w}$ is given in the form $\hat{\mathbf{w}} = \mathbf{M}^{-1}\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$ and its covariance matrix equals $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2 \mathbf{M}^{-1}$, where $\mathbf{M} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix for the design $\mathbf{X}$.

The introduction and the basic problems of weighing designs can be found in Jacroux et al. (1983), Sathe and Shenoy (1990) and the references given there. Among many questions regard to the weighing designs, different optimality criteria of such designs are considered. In present paper, we consider D-optimal designs. The design $\mathbf{X}_D$ is called D-optimal in the given class $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ if

$$\det\left(\mathbf{X}_D^{'}\mathbf{G}^{-1}\mathbf{X}_D\right)^{-1} = \min\left(\det\left(\mathbf{M}^{-1}\right) : \mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}\right).$$

If $\det\left(\mathbf{M}^{-1}\right)$ attains the lowest bound then the design is called regular D-optimal. In other case, such design is called D-optimal. Each regular D-optimal design is D-optimal whereas the inverse sentence may not be true. For a recent account of the theory of regular D-optimal chemical balance weighing designs we refer the reader to Masaro and Wong (2008), Katulska and Smaga (2013), Neubauer and Pace (2010) and Smaga (2014).

In any class $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ we are not able to determine regular D-optimal design, for example based on the methods given in Ceranka and Graczyk (2014, 2015). So, the basic idea of this paper is to determine regular D-optimal design in the classes in that it is impossible yet. For that reason, we give new construction method of regular D-optimal design based on the set of incidence matrices of the balanced bipartite weighing designs and the ternary balanced block designs. We remind the definition of the D-optimal design and the theorem determining the parameters of the regular D-optimal design given in Ceranka and Graczyk (2014, 2015).

Definition 1.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}\{-1, 0, 1\}$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is regular D-optimal if $\det(\mathbf{M}^{-1}) = \left(g(1-\rho)\, m^{-1}\right)^p$.

Theorem 1.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}\{-1, 0, 1\}$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1), is regular D-optimal if and only if

(i) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ for $\rho = 0$ and

(ii) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ and $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$ if $0 < \rho < 1$.

2. Constructions

Now, we remind the definitions and the properties of the balanced bipartite weighing design and the ternary balanced block design.

A balanced bipartite weighing design there is an arrangement of v treatments into b blocks in such a way that each block containing k_1 distinct treatments is divided into 2 subblocks containing k_{11} and k_{21} treatments, respectively, where $k_1 = k_{11} + k_{21}$. Each treatment appears in r_1 blocks, moreover each pair of treatments from different subblocks appears together in λ_{11} blocks and each pair of treatments from the same subblock appears together in λ_{21} blocks. The integers v, b_1, r_1, k_{11}, k_{21}, λ_{11}, λ_{21} are called the parameters of the balanced bipartite weighing design and satisfy the following equalities

$$vr_1 = b_1k_1, \qquad b = \frac{\lambda_{11}v(v-1)}{2k_{11}k_{21}}, \qquad \lambda_{21} = \frac{\lambda_{11}\left[k_{11}(k_{11}-1) + k_{21}(k_{21}-1)\right]}{2k_{11}k_{21}},$$

$$r_1 = \frac{\lambda_{11}k_1(v-1)}{2k_{11}k_{21}}.$$

$\mathbf{N}_1^*$ is the $v \times b_1$ incidence matrix of the balanced bipartite weighing design and $\mathbf{N}_1^*(\mathbf{N}_1^*)' = (r_1 - \lambda_{11} - \lambda_{21})\mathbf{I}_v + (\lambda_{11} + \lambda_{21})\mathbf{1}_v\mathbf{1}_v'$.

Any ternary balanced block design is a design that describe how to replace v treatments in b_2 blocks, each of size k_2 in such a way that each treatment appears 0, 1 or 2 times in r_2 blocks. Each of the distinct pairs of treatments appears λ_2 times. Any ternary balanced block design is regular, that is, each treatment occurs alone in ρ_{12} blocks and is repeated two times in ρ_{22} blocks, where ρ_{12} and ρ_{22} are constant for the design. It is straightforward to verify that $vr_2 = b_2 k_2$, $r = \rho_{12} + 2\rho_{22}$, $\lambda_2(v-1) = \rho_{12}(k_2 - 1) + 2\rho_{22}(k_2 - 2)$. $\mathbf{N}_2$ is the $v \times b_2$ incidence matrix of such a design with elements equal to 0, 1 or 2 and moreover $\mathbf{N}_2\mathbf{N}_2' = (\rho_{12} + 4\rho_{22} - \lambda_2)\mathbf{I}_v + \lambda_2\mathbf{1}_v\mathbf{1}_v'$.

Let $\mathbf{N}_1^*$ be the incidence matrix of the balanced bipartite weighing design with the parameters v, b_1, r_1, k_{11}, k_{21}, λ_{11}, λ_{21}. From $\mathbf{N}_1^*$ we form matrix $\mathbf{N}_1$ by replacing k_{11} unities equal to +1 of each column which correspond to the elements belonging to the first subblock by -1. Accordingly, each column of the matrix $\mathbf{N}_1$ will contain k_{11} elements equal to -1, k_{21} elements equal to 1 and $v - k_{11} - k_{21}$ elements equal to 0. Moreover, let $\mathbf{N}_2$ be the incidence matrix of the ternary balanced block design with the parameters v, b_2, r_2, k_2, λ_2, ρ_{12}, ρ_{22}. Thus

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' \\ \mathbf{N}_2' - \mathbf{1}_{b_2}\mathbf{1}_v' \end{bmatrix}. \tag{2.1}$$

Under assumption $k_{11} \neq k_{21}$, each column of $\mathbf{X}$ contains $r_{21} + \rho_{22}$ elements equal to 1, $r_{11} + b_2 - \rho_{12} - \rho_{22}$ elements equal -1 and $b_1 - r_1 + \rho_{12}$ elements equal 0, $r_1 = r_{11} + r_{21}$, where $r_{11} = \dfrac{\lambda_{11}(v-1)}{2k_{21}}$, $r_{21} = \dfrac{\lambda_{11}(v-1)}{2k_{11}}$. Moreover, $p = v$ $n = b_1 + b_2$, $m = r_1 + b_2 - \rho_{12}$.

Lemma 2.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}\{-1, 0, 1\}$ given in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1), is nonsingular if and only if $k_{11} \neq k_{21}$ or $v \neq k_2$.

Proof. Consider the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}\{-1, 0, 1\}$ in the form (2.1). We shall establish the Lemma if we give the determinant of the information matrix for the design. We have

$$\mathbf{X}'\mathbf{X} = \left(r_1 - \lambda_{21} + \lambda_{11} + r_2 + 2\rho_{22} - \lambda_2\right)\mathbf{I}_v + \left(\lambda_{21} - \lambda_{11} + b_2 - 2r_2 + \lambda_2\right)\mathbf{1}_v\mathbf{1}_v'$$

and

$$\det\left(\mathbf{X}'\mathbf{X}\right) = \left(r_1 - \lambda_{21} + \lambda_{11} + r_2 + 2\rho_{22} - \lambda_2\right)^{v-1}\left(\frac{r_2}{k_2}\left(v - k_2\right)^2 + \frac{r_1}{k_1}\left(k_{11} - k_{21}\right)^2\right).$$

Now, the Lemma is obvious.

2.1 Construction of D-optimal design for uncorrelated errors

Theorem 2.1. Let $\rho = 0$. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}\{-1, 0, 1\}$ in (2.2) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1), is regular D-optimal if and only if

$$\lambda_{21} - \lambda_{11} + b_2 - 2r_2 + \lambda_2 = 0. \tag{2.2}$$

Proof. It is sufficient to use Theorem 1.1 and Lemma 2.1 together with observation that the information matrix of the design is proportional to the identity matrix.

In particular, the equality in (2.2) is true, when $\lambda_{21} = \lambda_{11}$ and $b_2 = 2r_2 - \lambda_2$. Hence we have the following Corollary.

Corollary 2.1. Let $\rho = 0$. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}\{-1, 0, 1\}$ in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1), is regular D-optimal if and only if $\lambda_{21} = \lambda_{11}$ and $b_2 = 2r_2 - \lambda_2$.

Theorem 2.2. Let $\rho = 0$. The existence of the balanced bipartite weighing design and the ternary balanced block design with the parameters

(i) $v = s$, $b_1 = 0.5st(s-1)$, $r_1 = 2t(s-1)$, $k_{11} = 1$, $k_{21} = 3$, $\lambda_{11} = \lambda_{21} = 3t$ and $v = s$, $b_2 = ws$, $r_2 = w(s-u)$, $k_2 = s-u$, $\lambda_2 = w(s-2u)$, $\rho_{12} = w(s-u^2)$, $\rho_{22} = 0.5wu(u-1)$, $t, w = 1,2,\ldots$, $u = 2,3,4$, $s = u^2+1, u^2+2,\ldots$,

(ii) $v = 3s+1$, $b_1 = 0.5st(3s+1)$, $r_1 = 2st$, $k_{11} = 1$, $k_{21} = 3$, $\lambda_{11} = \lambda_{21} = t$ and $v = 3s+1$,

$b_2 = w(3s+1)$, $r_2 = w(3s-u+1)$, $k_2 = 3s-u+1$, $\lambda_2 = w(3s-2u+1)$, $\rho_{12} = w(3s-u^2+1)$, $\rho_{22} = 0.5wu(u-1)$, $s, t, w = 1,2,\ldots$, $u = 2,3,4$,

(iii) $v = 9t+1$, $b_1 = ut(9t+1)$, $r_1 = 9ut$, $k_{11} = 3$, $k_{21} = 6$, $\lambda_{11} = \lambda_{21} = 4u$ and $v = 9t+1$, $b_2 = z(9t+1)$, $r_2 = z(9t-w+1)$, $k_2 = 9t-w+1$, $\lambda_2 = z(9t-2w+1)$, $\rho_{12} = z(9t-w^2+1)$, $\rho_{22} = 0.5zw(w-1)$, $u, z = 1,2,\ldots$, if $w = 2,3$ then $t = 1,2,\ldots$, if $w = 4$ then $t = 2,3,\ldots$,

(iv) $v = 36t+1$, $b_1 = st(36t+1)$, $r_1 = 9st$, $k_{11} = 3$, $k_{21} = 6$, $\lambda_{11} = \lambda_{21} = s$ and $v = 36t+1$, $b_2 = u(36t+1)$, $r_2 = u(36t-w+1)$, $k_2 = 36t-w+1$, $\lambda_2 = u(36t-2w+1)$, $\rho_{12} = u(36t-w^2+1)$, $\rho_{22} = 0.5uw(w-1)$, $s, t, u = 1,2,\ldots$, $w = 2,3,4$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1).

Proof. It is easy to prove that the parameters of the balanced bipartite weighing design and the ternary balanced block design satisfy the condition (2.2).

The equality in (2.2) is also true if $\lambda_{21} - \lambda_{11} = \alpha$ and $b_2 + \lambda_2 - 2r_2 = -\alpha$, where α is any integer and $\alpha \neq 0$.

Theorem 2.3. Let $\rho = 0$. The existence of the balanced bipartite weighing design and the ternary balanced block design with the parameters

(i) $v = 2s$, $b_1 = 2s(2s-1)$, $r_1 = 7(2s-1)$, $k_{11} = 2$, $k_{21} = 5$, $\lambda_{11} = 20$, $\lambda_{21} = 22$ and $v = k_2 = 2s$, $b_2 = r_2 = 4s+u-2$, $\lambda_2 = 4(s-1)+u$, $\rho_{12} = u$, $\rho_{22} = 2s-1$, $s = 5,6,\ldots$, $u = 1,2,\ldots$,

(ii) $v = 2s + 1$, $b_1 = s(2s + 1)$, $r_1 = 7s$, $k_{11} = 2$, $k_{21} = 5$, $\lambda_{11} = 10$, $\lambda_{21} = 11$ and $v = k_2 = 2s + 1$, $b_2 = r_2 = 2s + u$, $\lambda_2 = 2s + u - 1$, $\rho_{12} = u$, $\rho_{22} = s$, $s = 4,5,7$, $u = 1,2,...$

(iii) $v = 4s + 1$, $b_1 = s(4s + 1)$, $r_1 = 5s$, $k_{11} = 1$, $k_{21} = 4$, $\lambda_{11} = 2$, $\lambda_{21} = 3$ and $v = k_2 = 4s + 1$, $b_2 = r_2 = 4s + u$, $\lambda_2 = 4s + u - 1$, $\rho_{12} = u$, $\rho_{22} = 2s$, $s = 1,2$, $u = 1,2,...$,

(iv) $v = 4s + 1$, $b_1 = s(4s + 1)$, $r_1 = 8s$, $k_{11} = 2$, $k_{21} = 6$, $\lambda_{11} = 6$, $\lambda_{21} = 8$ and $v = k_2 = 4s + 1$, $b_2 = r_2 = 8s + u + 1$, $\lambda_2 = 8s + u - 1$, $\rho_{12} = u + 1$, $\rho_{22} = 4s$, $u = 1,2,...$, $s = 2,3,...$,

(v) $v = b_1 = r_1 = 5$, $k_{11} = 1$, $k_{21} = 4$, $\lambda_{11} = 2$, $\lambda_{21} = 3$ and $v = 5$, $b_2 = 5(s + 1)$, $r_2 = 4(s + 1)$, $k_2 = 4$, $\lambda_2 = 3s + 2$, $\rho_{12} = 4s$, $\rho_{22} = 2$, $s = 1,2,...$,

(vi) $v = 5$, $b_1 = 10$, $r_1 = 6$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $v = 5$, $b_2 = 5(s + 2)$, $r_2 = 3(s + 2)$, $k_2 = 3$, $\lambda_2 = s + 3$, $\rho_{12} = s + 6$, $\rho_{22} = s$, $s = 1,2,...$,

(vii) $v = 6$, $b_1 = 30$, $r_1 = 15$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 4$, $\lambda_{21} = 2$ and $v = 6$, $b_2 = 2(s + 5)$, $r_2 = s + 5$, $k_2 = 3$, $\lambda_2 = 2$, $\rho_{12} = 5 - s$, $\rho_{22} = s$, $s = 1,2,3,4$,

(viii) $v = 11$, $b_1 = 11$, $r_1 = 6$, $k_{11} = 1$, $k_{21} = 5$, $\lambda_{11} = 1$, $\lambda_{21} = 2$ and $v = k_2 = 11$, $b_2 = r_2 = u + 10$, $\lambda_2 = u + 9$, $\rho_{12} = u$, $\rho_{22} = 5$, $u = 1,2,...$,

(ix) $v = 11$, $b_1 = 55$, $r_1 = 30$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = b_2 = 11$, $r_2 = k_2 = 7$, $\lambda_2 = 4$, $\rho_{12} = 5$, $\rho_{22} = 1$,

(x) $v = 12$, $b_1 = 66$, $r_1 = 33$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = 12$, $b_2 = 3(2s + 5)$, $r_2 = 2(2s + 5)$, $k_2 = 8$, $\lambda_2 = 2(s + 3)$, $\rho_{12} = 6 - 2s$, $\rho_{22} = 3s + 2$, $s = 1,2$,

(xi) $v = 15$, $b_1 = 105$, $r_1 = 21w$, $k_{11} = w$, $k_{21} = 2w$, $\lambda_{11} = 2w^2$, $\lambda_{21} = 0.5w(5w - 3)$ and $v = 15$, $b_2 = 3(s + 4)$, $r_2 = 2(s + 4)$, $k_2 = 10$, $\lambda_2 = s + 5$, $\rho_{12} = 6 - 2s$, $\rho_{22} = 2s + 1$, $s, w = 1,2$, implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1).

Proof. It follows immediately that the parameters given in (i)–(xi) satisfy the condition (2.2).

Theorem 2.4. Let $\rho = 0$. The existence of the balanced bipartite weighing design with the parameters

(i) $v = 9$, $b_1 = 18$, $r_1 = 10$, $k_{11} = 2$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 2$,

(ii) $v = 9$, $b_1 = 36$, $r_1 = 12$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$,

(iii) $v = 9$, $b_1 = 36$, $r_1 = 24$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$,

and the ternary balanced block design with the parameters $v = 9$, $b_2 = 3(s+4)$, $r_2 = 2(s+4)$, $k_2 = 6$, $\lambda_2 = s+5$, $\rho_{12} = 8$, $\rho_{22} = s$, $s = 1,2,...$, implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1).

Proof. Obviously, the parameters given above satisfy the condition (2.2).

2.2. Construction of D-optimal design for positive correlated errors

Theorem 2.5. Let $0 < \rho < 1$ and $k_{11} \neq k_{21}$. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1), is regular D-optimal if and only if (2.2) holds and

$$b_2 - r_2 + \frac{\lambda_{11}(v-1)(k_{11}-k_{21})}{2k_{11}k_{21}} = 0. \tag{2.3}$$

Proof. We begin by considering the design matrix $\mathbf{X}$ of (2.1). By Theorem 1.1(i) and from $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ we obtain the condition (2.2). The formula $\mathbf{X}'\mathbf{1}_n = 0$ given in Theorem 1.1(ii) indicates that in each column of $\mathbf{X}$ the number of elements equal to -1 is equal to the number of elements equal to 1. For that reason we obtain the equality $r_{11} + b_2 - \rho_{11} - \rho_{22} = r_{21} + \rho_{22}$. Furthermore, if $k_{11} \neq k_{21}$ then $r_{11} = \frac{\lambda_{11}(v-1)}{2k_{21}}$, $r_{21} = \frac{\lambda_{11}(v-1)}{2k_{11}}$. Now, from Condition (ii) of Theorem 1.1, we obtain (2.3).

Theorem 2.6. Let $0 < \rho < 1$. The existence of the balanced bipartite weighing design and the ternary balanced block design with the parameters

(i) $v = 3s+1$, $b_1 = 0.5ust(3s+1)$, $r_1 = 2ust$, $k_{11} = 1$, $k_{21} = 3$, $\lambda_{11} = \lambda_{21} = ut$ and $v = 3s+1$, $b_2 = st(3s+1)$, $r_2 = st(3s-w+1)$, $k_2 = 3s-w+1$, $\lambda_2 = st(3s-2w+1)$, $\rho_{12} = st(3s-w^2+1)$, $\rho_{22} = 0.5stw(w-1)$, $t, u = 1,2,\ldots$, if $w = 2,3$ then $s = w, w+1,\ldots$, if $w = 4$ then $s = w+2, w+3,\ldots$,

(ii) $v = 3(s+1)$, $b_1 = 0.5(3t(s+1)(3s+2))$, $r_1 = 2t(3s+2)$, $k_{11} = 1$, $k_{21} = 3$, $\lambda_{11} = \lambda_{21} = 3t$ and $v = 3(s+1)$, $b_2 = t(s+1)(3s+2)$, $r_2 = st(3s+2)$, $k_2 = 3s$, $\lambda_2 = t(s-1)(3s+2)$, $\rho_{12} = t(s-2)(3s+2)$, $\rho_{22} = t(3s+2)$, $t = 1,2,\ldots$, $s = 2,3,\ldots$,

(iii) $v = us+1$, $b_1 = 0.5ust(us+1)$, $r_1 = 2ust$, $k_{11} = 1$, $k_{21} = 3$, $\lambda_{11} = \lambda_{21} = 3t$ and $v = us+1$, $b_2 = st(us+1)$, $r_2 = st(u(s-1)+1)$, $k_2 = u(s-1)+1$, $\lambda_2 = st(u(s-2)+1)$, $\rho_{12} = st(u(s-u)+1)$, $\rho_{22} = 0.5stu(u-1)$, $t = 1,2,\ldots$, $u = 2,3,4$, $s = u, u+1,\ldots$,

(iv) $v = b_1 = r_1 = 5$, $k_{11} = 1$, $k_{21} = 4$, $\lambda_{11} = 2$, $\lambda_{21} = 3$ and $v = 5$, $b_2 = 15$, $r_2 = 12$, $k_2 = 4$, $\lambda_2 = \rho_{12} = 8$, $\rho_{22} = 2$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}\{-1, 0, 1\}$ in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1).

Proof. It follows easily that (i) – (iv) are the parameters of the regular D-optimal design.

Theorem 2.7. Let $0 < \rho < 1$. The existence of the balanced bipartite weighing design with the parameters $v = 9t+1$, $b_1 = tuw(9t+1)$, $r_1 = 9tuw$, $k_{11} = 3$,

$k_{21} = 6$, $\lambda_{11} = \lambda_{21} = 4uw$, $w = 1,2,4$, and the ternary balanced block design with the parameters

(i) for $w = 1$, $v = 9t + 1$, $b_2 = ut(9t + 1)$, $r_2 = ut(9t - 2)$, $k_2 = 9t - 2$, $\lambda_2 = ut(9t - 5)$, $\rho_{12} = ut(9t - 8)$, $\rho_{22} = 3ut$, $t, u = 1,2,\ldots$,

(ii) for $w = 2$, $v = 9t + 1$, $b_2 = 3ut(9t + 1)$, $r_2 = 3ut(9t - 1)$, $k_2 = 9t - 1$, $\lambda_2 = \rho_{12} = 9ut(3t - 1)$, $\rho_{22} = 3ut$, $t, u = 1,2,\ldots$,

(iii) for $w = 4$, $v = 9t + 1$, $b_2 = 3ut(9t + 1)$, $r_2 = 9ut(3t - 1)$, $k_2 = 3(3t - 1)$, $\lambda_2 = 3ut(9t - 7)$, $\rho_{12} = 9ut(3t - 5)$, $\rho_{22} = 18ut$, $u = 1,2,\ldots$, $t = 2,3,\ldots$

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}\{-1, 0, 1\}$ in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1).

Proof. It is evident that (i) – (iii) are the parameters of the regular D-optimal design.

Theorem 2.8. Let $0 < \rho < 1$. The existence of the balanced bipartite weighing design with the parameters $v = 36t + 1$, $b_1 = stw(36t + 1)$, $r_1 = 9stw$, $k_{11} = 3$, $k_{21} = 6$, $\lambda_{11} = \lambda_{21} = sw$, $w = 1,2,4$, and the ternary balanced block design with the parameters

(i) for $w = 1$, $v = 36t + 1$, $b_1 = st(36t + 1)$, $r_1 = 9st$, $k_{11} = 3$, $k_{21} = 6$, $\lambda_{11} = \lambda_{21} = s$ and $v = 36t + 1$, $b_2 = st(36t + 1)$, $r_2 = 2st(18t - 1)$, $k_2 = 2(18t - 1)$, $\lambda_2 = 3st(12t - 1)$, $\rho_{12} = 4st(9t - 2)$, $\rho_{22} = 3st$, $s, t = 1,2,\ldots$,

(ii) for $w = 2$, $v = 36t + 1$, $b_1 = 2st(36t + 1)$, $r_1 = 18st$, $k_{11} = 3$, $k_{21} = 6$, $\lambda_{11} = \lambda_{21} = 2s$ and $v = 36t + 1$, $b_2 = 3st(36t + 1)$, $r_2 = 3st(36t - 1)$, $k_2 = 36t - 1$, $\lambda_2 = 9st(12t - 1)$, $\rho_{12} = 9st(12t - 1)$, $\rho_{22} = 3st$, $s, t = 1,2,\ldots$,

(iii) for $w = 4$, $v = 36t + 1$, $b_1 = 4st(36t + 1)$, $r_1 = 36st$, $k_{11} = 3$, $k_{21} = 6$, $\lambda_{11} = \lambda_{21} = 4s$ and $v = 36t + 1$, $b_2 = 3st(36t + 1)$, $r_2 = 9st(12t - 1)$, $k_2 = 3(12t - 1)$, $\lambda_2 = 3st(36t - 7)$, $\rho_{12} = 9st(12t - 5)$, $\rho_{22} = 18st$, $s, t = 1,2,\ldots$, implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ in (2.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1).

Proof. It is evident that (i) – (iii) are the parameters of the regular D-optimal design.

3. Examples

Example 3.1

Let us consider the experiment in that we determine unknown measurements of $p = 5$ objects and $n = 25$ measurements under assumption that the measurement errors are uncorrelated. According to the Theorem 2.3 (vi) we consider the balanced bipartite weighing design with the parameters $v = 5$, $b_1 = 10$, $r_1 = 6$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and the incidence matrix $\mathbf{N}_1$ and ternary balanced block design with the parameters $v = 5$, $b_2 = 15$, $r_2 = 9$, $k_2 = 3$, $\lambda_2 = 4$, $\rho_{12} = 7$, $\rho_{22} = 1$ and incidence matrix $\mathbf{N}_2$, where

$$\mathbf{N}_1 = \begin{bmatrix} 1_2 & 1_2 & 1_1 & 0 & 0 \\ 1_2 & 0 & 1_2 & 1_1 & 0 \\ 1_2 & 0 & 0 & 1_2 & 1_1 \\ 1_2 & 1_1 & 0 & 0 & 1_2 \\ 0 & 1_2 & 1_2 & 0 & 1_1 \\ 1_1 & 1_2 & 0 & 1_2 & 0 \\ 0 & 1_2 & 0 & 1_1 & 1_2 \\ 0 & 1_1 & 1_2 & 1_2 & 0 \\ 1_1 & 0 & 1_2 & 0 & 1_2 \\ 0 & 0 & 1_1 & 1_2 & 1_2 \end{bmatrix},$$

$$\mathbf{N}_2' = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

According to the theory in Section 2 we form matrix $\mathbf{X} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix}$ which is regular D-optimal chemical balance weighing design in the class $\mathbf{\Phi}_{25\times 5,14}\{-1, 0, 1\}$, where

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & -1 & -1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 \\ -1 & -1 & 0 & 1 & -1 \\ -1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 \end{bmatrix}.$$

Example 3.2

Let us consider the experiment in that we determine unknown measurements of $p = 7$ objects and $n = 14$ measurements assuming that the errors are positive correlated. According to the Theorem 2.6 (ii) $(w = s = 2, t = u = 1)$ we consider the balanced bipartite weighing design with the parameters $v = 7$, $b_1 = 7$, $r_1 = 4$, $k_{11} = 1$, $k_{21} = 3$, $\lambda_{11} = 1$, $\lambda_{21} = 1$ and the incidence matrix $\mathbf{N}_1$ and the ternary balanced block design with the parameters $v = 7$, $b_2 = 7$, $r_2 = 5$, $k_2 = 5$, $\lambda_2 = 3$, $\rho_{12} = 3$, $\rho_{22} = 1$ and incidence matrix $\mathbf{N}_2$, where

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 0 \\ 0 & 0 & 1_2 & 1_1 & 1_2 & 0 & 1_2 \\ 1_1 & 0 & 0 & 1_2 & 1_2 & 1_2 & 0 \\ 0 & 1_2 & 0 & 0 & 1_1 & 1_2 & 1_2 \\ 1_2 & 0 & 1_2 & 0 & 0 & 1_2 & 1_1 \\ 1_2 & 1_1 & 0 & 1_2 & 0 & 0 & 1_2 \\ 1_2 & 1_2 & 1_1 & 0 & 1_2 & 0 & 0 \end{bmatrix} \quad \mathbf{N}_2 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 2 \end{bmatrix}.$$

Thus

$$\mathbf{X}' = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & -1 \\ 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

is regular D-optimal chemical balance weighing design in the class $\mathbf{\Phi}_{14\times 7,8}\{-1, 0, 1\}$.

4. Discussion

It is worth mentioning that the regular D-optimal design is determined under appropriate assumptions regard to the measuring errors and in the given class of matrices having elements equal to $-1, 0, 1$ and, that's more, for given number of objects p and measurements n. For example, under assumption that the errors are uncorrelated, based on Ceranka and Graczyk (2014), we are not able to determine regular D-optimal design for $p = 6$ and $p = 8$ objects. In contrast, based on present method, we determine optimal design in the class for 6 objects and 21 measurements and for the class for 8 objects and 36 measurements. On the other hand, when the measurements are equally positive correlated, based on Ceranka and Graczyk (2015), we determine regular D-optimal design in the class of weighing matrices for $p = 7$ objects and $n = 35$ measurements. By used of the construction presented above, regular D-optimal design exists in the class for 7 objects and 14 measurements. As you can see, the list of classes in that regular D-optimal chemical balance weighing design exists is not closed.

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