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REGULAR D-OPTIMAL WEIGHING DESIGNS WITH NON-NEGATIVE CORRELATIONS OF ERRORS CONSTRUCTED FROM SOME BLOCK DESIGNS

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Summary

In this paper, the issues related to the regular D-optimal chemical balance weighing design are considered. We study these designs under assumption: the measurement errors are equally non-negative correlated they have the same variances. Here we present the existence conditions of the regular D-optimal design and construction methods.

Keywords and phrases: balanced bipartite weighing design, balanced incomplete block design, chemical balance weighing design, D-optimality

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1. Introduction

In some problems related to the determining unknown measurements of objects the following linear model is considered

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$$

(i) $\mathbf{y}$ is an $n\times 1$ vector of observed results of n measurements,

(ii) $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$, $\mathbf{\Phi}_{n\times p}(-1,0,1)$ is the class of $n\times p$ $(n\geq p)$ matrices $\mathbf{X}=(x_{ij})$, $i=1,2,...,n$, $j=1,2,...,p$, of known elements equal to -1, 1 or 0,

(iii) $\mathbf{w}$ is a $p\times 1$ vector of unknown measurements of p objects,

(iv) $\mathbf{e}$ is an $n\times 1$ vector of random errors.

We assume that $\mathrm{E}(\mathbf{e})=\mathbf{0}_n$ and $\mathrm{Cov}(\mathbf{e})=\sigma^2\mathbf{G}$, where $\mathbf{0}_n$ is vector of zeros, $\sigma>0$ is known parameter, $\mathbf{G}$ is the $n\times n$ symmetric positive definite matrix of known elements given in the form

$$\mathbf{G}=g\left((1-\rho)\mathbf{I}_n+\rho\mathbf{1}_n\mathbf{1}_n^{'}\right),\quad g>0,\ 0\leq\rho<1, \tag{1.1}$$

where g,ρ are known, $\mathbf{I}_n$ denotes identity matrix of rank n and $\mathbf{1}_n$ denotes $n\times 1$ vector of ones. The inverse of matrix $\mathbf{G}$ is given as $\mathbf{G}^{-1}=(g(1-\rho))^{-1}\left[\mathbf{I}_n-\dfrac{\rho}{1+\rho(n-1)}\mathbf{1}_n\mathbf{1}_n^{'}\right]$.

For the estimation of unknown measurements of objects $\mathbf{w}$ we use the normal equations $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}}=\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$. The design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ is nonsingular if and only if the matrix $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, i.e. if and only if $\mathbf{X}$ is of full column rank. Assuming that $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, the generalized least squares estimator of $\mathbf{w}$ is given by $\hat{\mathbf{w}}=\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$ and $\mathrm{Var}(\hat{\mathbf{w}})=\sigma^2\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$. The matrix $\mathbf{M}=\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix of the design $\mathbf{X}$.

The problem is to estimate all unknown measurements of objects with the smallest product of their variances and because of this D-optimality criterion is considered. Therefore, for each form of $\mathbf{G}$ and given number of measurements n and number of objects p, in the set of all design matrices (interchangeable called "designs") $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ we indicate such matrix $\mathbf{X}$ for which the determinant of $\mathbf{M}^{-1}$ is minimal. For details, we refer Jacroux et al. (1983), Masaro and Wong (2008), Katulska and Smaga (2013). Selected applications of such designs are given in Banerjee (1975). The relations between chemical balance weighing designs and the factorial designs are given in Cheng et al. (2004). The practical applications of such designs in the agriculture, medicine as well as in industry branches are presented in Bose and Bagchi (2007), Jacroux (2009).

The aim of the paper is to determine the D-optimal chemical balance weighing designs in the classes in that it was not possible according to the construction methods given in previous papers, see Ceranka and Graczyk (2014-2016). Here, we present new results related to the D-optimal chemical balance weighing designs assuming that the random errors are equally non-negative correlated and with the same variances. Presented construction method of D-optimal design is based on the incidence matrices of the balanced incomplete block designs and balanced bipartite weighing designs. We give the lower bound for the determinant of the inverse of the information matrix and the list of the parameters of D-optimal experimental plans.

2. The main result

Let us consider the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (1.1). From Ceranka and Graczyk (2016) we have

Definition 2.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (1.1), is regular D-optimal if $\det(\mathbf{M}^{-1}) = \left(g(1-\rho)m^{-1}\right)^p$, where $m = \max\{m_1, m_2, ..., m_p\}$, m_j represents the number of elements equal to -1 and 1 in j^{th} column of $\mathbf{X}$, the number of nonzero elements, $j = 1,2,..., p$.

Theorem 2.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (1.1), is regular D-optimal if and only if

(i) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ if $\rho = 0$,

(ii) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$ and $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$ if $0 < \rho < 1$.

It worth noting, the condition $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$ indicates that the numbers of elements equal to –1 and +1 in each column of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ are the same.

The aim of the paper is to determine D-optimal chemical balance weighing design in given class $\mathbf{\Phi}_{n\times p}(-1,0,1)$, i.e. to give the design matrix. Many different block designs are used in the literature for the construction of such design matrix. Here, we introduce the construction of the matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ from the incidence matrices of the balanced incomplete block design and the balanced bipartite weighing design. Thus, we recall the definitions of these designs.

Balanced incomplete block design there is the design, which describes how to replace v treatments in b blocks such that each block containing k distinct treatments and each treatment appears in r blocks. Every pair of treatments appears in λ blocks. The parameters v, b, r, k, λ are related by the following identities $vr = bk$, $\lambda(v-1) = r(k-1)$. Let $\mathbf{N}$ be the incidence matrix of such design with elements equal to 1 or 0. We have $\mathbf{N}\mathbf{N}' = (r-\lambda)\mathbf{I}_v + \lambda \mathbf{1}_v\mathbf{1}_v'$.

The balanced bipartite weighing designs there is the design in that we replace v treatments in b blocks such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, $k = k_1 + k_2$. Each treatment appears in r blocks. Every pair of treatments from different subblocks appears together in λ_1 blocks and every pair of treatments from the same subblock appears together in λ_2 blocks. The parameters are related by the following identities

$vr = b(k_1+k_2)$, $b = 0.5\lambda_1 v(v-1)(k_1k_2)^{-1}$, $r = 0.5\lambda_1(k_1+k_2)(v-1)(k_1k_2)^{-1}$,

$\lambda_2 = 0.5(\lambda_1(k_1(k_1-1)+k_2(k_2-1)))(k_1k_2)^{-1}$. If $k_1 \neq k_2$, then each object occurs in r_1 blocks in the first subblock and in r_2 blocks in the second subblock, $r_1 + r_2 = r$, $r_1 = 0.5\lambda_1(v-1)k_2^{-1}$, $r_2 = 0.5\lambda_1(v-1)k_1^{-1}$. $\mathbf{N}^*$ is the incidence matrix of such design with the elements equal to 0 or 1 and $\mathbf{N}^*\mathbf{N}^{*'} = (r-\lambda_1-\lambda_2)\mathbf{I}_v + (\lambda_1+\lambda_2)\mathbf{1}_v\mathbf{1}_v'$.

Let us consider the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form

$$\mathbf{X} = \begin{bmatrix} 2\mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'} \\ \mathbf{N}_2^{'} \end{bmatrix}, \tag{2.1}$$

where $\mathbf{N}_1$ is the incidence matrix of the balanced incomplete block design with the parameters $v, b_1, r_1, k_1, \lambda_1$ and $\mathbf{N}_2^*$ is the incidence matrix of the balanced bipartite weighing design with the parameters $v, b_2, r_2, k_{12}, k_{22}, \lambda_{12}, \lambda_{22}$. From the matrix $\mathbf{N}_2^*$, we obtain matrix $\mathbf{N}_2$ by multiplying all elements belonging to the first subblock by –1. In the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1), $m = b_1 + r_2$, $n = b_1 + b_2$, $p = v$. If $k_{12} \neq k_{22}$, then each column of this matrix contains $r_1 + r_{22}$ elements equal +1, $b_1 - r_1 + r_{12}$ elements equal –1 and $b_2 + r_2$ elements equal to 0.

Lemma 2.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (1.1), is nonsingular if and only if

(i) $v \neq 2k_1$ or

(ii) $k_{12} \neq k_{22}$.

Proof. For the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given above we have

$$\mathbf{X}^{'}\mathbf{X} = \left(4(r_1 - \lambda_1) + r_2 - \lambda_{22} + \lambda_{12}\right)\mathbf{I}_v + \left(b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12}\right)\mathbf{1}_v\mathbf{1}_v^{'} \tag{2.2}$$

and

$$\det\left(\mathbf{X}^{'}\mathbf{X}\right) = \left(4(r_1 - \lambda_1) + r_2 - \lambda_{22} + \lambda_{12}\right)^{v-1} \cdot \left(\frac{r_1}{k_1}(v - 2k_1)^2 + \frac{r_2}{k_2}(k_{12} - k_{22})^2\right)$$

From the above, the thesis is obvious.

Theorem 2.2. Let $\rho = 0$. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (1.1), is regular D-optimal if and only if

$$b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12} = 0 . \tag{2.3}$$

Proof. The thesis is the result derived from the Theorem 2.1 and equality (2.2). In this case $\det(\mathbf{M}^{-1}) = (gm^{-1})^p$.

In particular, the equality given in (2.3) is true if $b_1 = 4(r_1 - \lambda_1)$ and $\lambda_{22} = \lambda_{12}$. Hence

Corollary 2.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$ is regular D-optimal if and only if

(i) $b_1 = 4(r_1 - \lambda_1)$ and

(ii) $\lambda_{22} = \lambda_{12}$.

Theorem 2.3. The existence of the balanced incomplete block designs and the balanced bipartite weighing design with the parameters

(i) $v = b_1 = 4s^2$, $r_1 = k_1 = s(2s+1)$, $\lambda_1 = s(s+1)$ and $v = 4s^2$, $b_2 = 2us^2(4s^2-1)$, $r_2 = 2u(4s^2-1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = \lambda_{22} = 3u$, $s,u = 1,2,\ldots$,

(ii) $v = b_1 = 4s^2$, $r_1 = k_1 = s(2s-1)$, $\lambda_1 = s(s-1)$ and $v = 4s^2$, $b_2 = 2us^2(4s^2-1)$, $r_2 = 2u(4s^2-1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = \lambda_{22} = 3u$, $s,u = 1,2,\ldots$,

(iii) $v = 4s^2$, $b_1 = 4st$, $r_1 = (2s-1)t$, $k_1 = s(2s-1)$ $\lambda_1 = (s-1)t$ and $v = 4s^2$, $b_2 = 2us^2(4s^2-1)$, $r_2 = 2u(4s^2-1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = \lambda_{22} = 3u$, $s,t,u = 1,2,\ldots$, $s \le t$,

(iv) $v=(2s-1)^2$, $b_1=4(s-1)t$, $r_1=4t(s-1)$, $k_1=2s-1$ $\lambda_1=(2s-3)t$ and $v=(2s-1)^2$, $b_2=2su(2s-1)^2(s-1)$, $r_2=8su(s-1)$, $k_{12}=1$, $k_{22}=3$, $\lambda_{12}=\lambda_{22}=3u$, $s,t,u=1,2,...$, $2s-1\leq 4t$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$.

Proof. It is easy to prove that parameters of the balanced incomplete block designs and the balanced bipartite weighing design satisfy the condition (2.3).

The equality given in (2.3) is also true if $b_1-4(r_1-\lambda_1)=\alpha$ and $\lambda_{22}-\lambda_{12}=-\alpha$, $\alpha\neq 0$. Hence

Theorem 2.4. The existence of the balanced incomplete block designs and the balanced bipartite weighing design with the parameters

(i) $v=4s+1$, $b_1=2(4s+1)$, $r_1=4s$, $k_1=2s$, $\lambda_1=2s-1$ and $v=4s+1$, $b_2=s(4s+1)$, $r_2=8s$, $k_{12}=2$, $k_{22}=6$, $\lambda_{12}=6$, $\lambda_{22}=8$ $s=2,3,...$, $4s+1$ is prime or prime power,

(ii) $v=4(s+1)$, $b_1=2(4s+3)$, $r_1=4s+3$, $k_1=2(s+1)$, $\lambda_1=2s+1$ and $v=4(s+1)$, $b_2=4(s+1)(4s+3)$, $r_2=7(4s+3)$, $k_{12}=2$, $k_{22}=5$, $\lambda_{12}=20$, $\lambda_{22}=22$, $s=1,2,...$,

(iii) $v=b_1=8s+7$, $r_1=k_1=4s+3$, $\lambda_1=2s+1$ and $v=8s+7$, $b_2=(4s+3)(8s+7)$, $r_2=7(4s+3)$, $k_{12}=2$, $k_{22}=5$, $\lambda_{12}=10$, $\lambda_{22}=11$, $s=1,2,...$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$.

Proof. It is easy to prove that parameters of the balanced incomplete block designs and the balanced bipartite weighing design satisfy the condition (2.3).

Theorem 2.5. Let $v=6$. The existence of the balanced incomplete block designs and the balanced bipartite weighing design with the parameters

(i) $b_1=10$, $r_1=5$, $k_1=3$, $\lambda_1=2$ and $b_2=6$, $r_2=6$, $k_{12}=1$, $k_{22}=5$, $\lambda_{12}=2$, $\lambda_{22}=4$,

(ii) $b_1 = 20$, $r_1 = 10$, $k_1 = 3$, $\lambda_1 = 4$ and $b_2 = 30$, $r_2 = 25$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 4$,

(iii) $b_1 = 30$, $r_1 = 10$, $k_1 = 2$, $\lambda_1 = 2$ and $b_2 = 6$, $r_2 = 6$, $k_{12} = 1$, $k_{22} = 5$ $\lambda_{12} = 2$, $\lambda_{22} = 4$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$.

Proof. It is easy to check that parameters of the balanced incomplete block designs and the balanced bipartite weighing design satisfy the condition (2.3).

Theorem 2.6. Let $v = 7$. The existence of the balanced incomplete block designs and the balanced bipartite weighing design with the parameters

(i) $b_1 = 21$, $r_1 = 6$, $k_1 = 2$, $\lambda_1 = 1$ and

(a) $b_2 = 21$, $r_2 = 9s$, $k_{12} = s$, $k_{22} = 2s$, $\lambda_{12} = 2s^2$, $\lambda_{22} = 2s^2 - 1$, $s = 1, 2$,

(b) $b_2 = 7$, $r_2 = 7$, $k_{12} = 3$, $k_{22} = 4$, $\lambda_{12} = 4$, $\lambda_{22} = 3$,

(ii) $b_1 = 7s$, $r_1 = 3s$, $k_1 = 3$, $\lambda_1 = s$, $s = 1,2,...,5$ and

(a) for $s = 1$, $b_2 = 21$, $r_2 = 21$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$,

(b) for $s = 2$, $b_2 = 21$, $r_2 = 15$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 4$, $\lambda_{22} = 6$,

(c) for $s = 3$, $b_2 = 7$, $r_2 = 7$, $k_{12} = 1$, $k_{22} = 6$, $\lambda_{12} = 2$, $\lambda_{22} = 5$,

(d) for $s = 4$, $b_2 = 42$, $r_2 = 15$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 12$,

(e) for $s = 5$, $b_2 = 21$, $r_2 = 18$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 5$, $\lambda_{22} = 10$

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$.

Proof. It is obvious that parameters of the designs given above satisfy the condition (2.3).

Theorem 2.7. Let $v = 8$. The existence of the balanced incomplete block designs with the parameters $b_1 = 28$, $r_1 = 7s$, $k_1 = 2s$, $\lambda_1 = s(2s-1)$ and the balanced bipartite weighing design with the parameters

(i) for $s = 1$

(a) $b_2 = 56$, $r_2 = 8$, $k_{12} = 2$, $k_{22} = 3$, $\lambda_{12} = 12$, $\lambda_{22} = 8$,

(b) $b_2 = 56$, $r_2 = 56$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 30$, $\lambda_{22} = 26$,

(ii) for $s = 2$

(a) $b_2 = 8$, $r_2 = 8$, $k_{12} = 1$, $k_{22} = 7$, $\lambda_{12} = 2$, $\lambda_{22} = 6$,

(b) $b_2 = 28$, $r_2 = 28$, $k_{12} = 2$, $k_{22} = 6$, $\lambda_{12} = 12$, $\lambda_{22} = 16$,

(c) $b_2 = 56$, $r_2 = 35$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 12$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$.

Proof. In order to prove this Theorem one may observe that parameters of the balanced incomplete block designs and the balanced bipartite weighing design satisfy the condition (2.3).

Theorem 2.8. Let $v = 10$. The existence of the balanced incomplete block designs with the parameters $b_1 = 30$, $r_1 = 3s$, $k_1 = s$, $\lambda_1 = s - 1$ and the balanced bipartite weighing design with the parameters

(i) for $s = 3$

(a) $b_2 = 90$, $r_2 = 27$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 4$, $\lambda_{22} = 2$

(b) $b_2 = 30$, $r_2 = 21$, $k_{12} = 3$, $k_{22} = 7$, $\lambda_{12} = 8$, $\lambda_{22} = 6$,

(ii) for $s = 4$

(a) $b_2 = 30$, $r_2 = 21$, $k_{12} = 1$, $k_{22} = 6$, $\lambda_{12} = 4$, $\lambda_{22} = 10$

(b) $b_2 = 10$, $r_2 = 10$, $k_{12} = 1$, $k_{22} = 9$, $\lambda_{12} = 2$, $\lambda_{22} = 8$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$.

Proof. It is easy to prove that parameters of the balanced incomplete block designs and the balanced bipartite weighing design satisfy the condition (2.3).

Theorem 2.9. Let $v = 11$. The existence of the balanced incomplete block designs with the parameters $b_1 = 11s$, $r_1 = 5s$, $k_1 = 5$, $\lambda_1 = 2s$ and the balanced bipartite weighing design with the parameters

(i) for $s = 1$

(a) $b_2 = 11$, $r_2 = 6$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 1$, $\lambda_{22} = 2$,

(b) $b_2 = 55$, $r_2 = 35$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$,

(ii) for $s = 2$, $b_2 = 55$, $r_2 = 25$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 4$, $\lambda_{22} = 6$,

(iii) for $s = 3$, $b_2 = 55$, $r_2 = 50$, $k_{12} = 3$, $k_{22} = 7$, $\lambda_{12} = 21$, $\lambda_{22} = 24$

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2 \mathbf{I}_n$.

Proof. It is enough to observe that parameters of the balanced incomplete block designs and the balanced bipartite weighing design satisfy the condition (2.3).

Theorem 2.10. Let $v = 13$. The existence of the balanced incomplete block designs and the balanced bipartite weighing design with the parameters

(i) $b_1 = 13$, $r_1 = 4$, $k_1 = 4$, $\lambda_1 = 1$ and

(a) $b_2 = 78$, $r_2 = 18s$, $k_{12} = s$, $k_{22} = 2s$, $\lambda_{12} = 2s^2$, $\lambda_{22} = 2.5(5s-3)$, $s = 1,2$

(b) $b_2 = 13s$, $r_2 = s + 12$, $k_{12} = \lambda_{22} = 5 - s$, $k_{22} = \lambda_{12} = 6 - s$, $s = 2,3$

(c) $b_2 = 26$, $r_2 = 20$, $k_{12} = 4$, $k_{22} = 6$, $\lambda_{12} = 8$, $\lambda_{22} = 7$,

(d) $b_2 = 78$, $r_2 = 66$, $k_{12} = 4$, $k_{22} = 7$, $\lambda_{12} = 28$, $\lambda_{22} = 27$,

(e) $b_2 = 39$, $r_2 = 39$, $k_{12} = 5$, $k_{22} = 8$, $\lambda_{12} = 20$, $\lambda_{22} = 19$

(ii) $b_1 = 26$, $r_1 = 8$, $k_1 = 4$, $\lambda_1 = 2$ and

(a) $b_2 = 39s$, $r_2 = 3(s+2)^2$, $k_{12} = 2s - s^2 + 3$, $k_{22} = 5$, $\lambda_{12} = 5(s+1)$, $\lambda_{22} = 5s + 3$,

(b) $b_2 = 26$, $r_2 = 26$, $k_{12} = 6$, $k_{22} = 7$, $\lambda_{12} = 14$, $\lambda_{22} = 12$,

(iii) $b_1 = 39$, $r_1 = 15$, $k_1 = 5$, $\lambda_1 = 5$ and

(a) $b_2 = 39$, $r_2 = 15$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$,

(b) $b_2 = 39s$, $r_2 = 6(s+5)$, $k_{12} = 2(3-s)$, $k_{22} = 11 - 3s$, $\lambda_{12} = 2(11-3s)$, $\lambda_{22} = 23 - 6s$, $s = 1,2$,

(c) $b_2 = 13s$, $r_2 = 7s + 6$, $k_{12} = 5 - s$, $k_{22} = s^{-1}(s^2 + 2s + 6)$, $\lambda_{12} = s + 5$, $\lambda_{22} = s + 6$, $s = 1,2$,

(iv) $b_1 = 26$, $r_1 = 12$, $k_1 = 6$, $\lambda_1 = 5$ and

(a) $b_2 = 39$, $r_2 = 24$, $k_{12} = 2$, $k_{22} = 6$, $\lambda_{12} = 6$, $\lambda_{22} = 8$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$.

Proof. It is obvious that parameters of the designs (i)-(iv) satisfy the condition (2.3).

Theorem 2.11. The existence of the balanced incomplete block designs and the balanced bipartite weighing design with the parameters

(i) $v = 9$, $b_1 = 18$, $r_1 = 8$, $k_1 = 4$, $\lambda_1 = 3$ and $v = 9$, $b_2 = 18$, $r_2 = 16$, $k_{12} = 2$, $k_{22} = 6$, $\lambda_{12} = 6$, $\lambda_{22} = 8$

(ii) $v = 12$, $b_1 = 11s$, $r_1 = 11$, $k_1 = 12s^{-1}$, $\lambda_1 = 12s^{-1} - 1$ and

(a) for $s = 2$, $b_2 = 132$, $r_2 = 132$, $k_{12} = 5$, $k_{22} = 7$, $\lambda_{12} = 70$, $\lambda_{22} = 62$,

(b) for $s = 3$, $b_2 = 66$, $r_2 = 33$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$,

(c) for $s = 4$, $b_2 = 132$, $r_2 = 77$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 20$, $\lambda_{22} = 22$,

(iii) $v=15$, $b_1=42$, $r_1=14$, $k_1=5$, $\lambda_1=4$ and $v=15$, $b_2=105$, $r_2=56$, $k_{12}=3$, $k_{22}=8$, $\lambda_{12}=15$, $\lambda_{22}=13$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{I}_n$.

Proof. Putting the parameters given above into the conditions (i) and (ii) of Corollary 2.1 you obtain the equality.

In the special case, if $r_1=\lambda_1$ then (2.3) implies that $b_1=\lambda_{12}-\lambda_{22}$ and

Corollary 2.2. For any v, the existence of the balanced bipartite weighing design with the parameters v, b_2, r_2, k_{12}, k_{22}, λ_{12}, λ_{22}, for which $k_{12}\neq k_{22}$ and $\lambda_{12}>\lambda_{22}$ implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form

$$\mathbf{X}=\begin{bmatrix}\mathbf{1}_{b_1}\mathbf{1}_v^{'}\\ \mathbf{N}_2^{'}\end{bmatrix}, \tag{2.4}$$

with the variance matrix of errors $\sigma^2\mathbf{I}_n$, where $b_1=\lambda_{12}-\lambda_{22}$.

Theorem 2.12. The existence of the balanced bipartite weighing design with the parameters

(i) $v=2s+1$, $b_2=s(2s+1)$, $r_2=8s$, $k_{12}=3$, $k_{22}=5$, $\lambda_{12}=15$, $\lambda_{22}=13$, $s=4,5,\ldots$,

(ii) $v=6s$, $b_2=6s(6s-1)$, $r_2=3(6s-1)$, $k_{12}=1$, $k_{22}=2$, $\lambda_{12}=4$, $\lambda_{22}=2$, $s=1,2,\ldots$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.4) with $b_1=2$ the variance matrix of errors $\sigma^2\mathbf{I}_{b_2+2}$.

Proof. The proof is straightforward by the observation that parameters of the designs (i)-(ii) satisfy the condition (2.3).

Theorem 2.13. The existence of the balanced bipartite weighing design with the parameters

(i) $v = 2s+1$, $b_2 = s(2s+1)$, $r_2 = 3s$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$, $s = 2,3,...$, except $s = 5$,

(ii) $v = 4s+1$, $b_2 = s(4s+1)$, $r_2 = 5s$, $k_{12} = 2$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 2$, $s = 2,3,...$,

(iii) $v = 2s+1$, $b_2 = s(2s+1)$, $r_2 = 6s$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 3,4,...$,

(iv) $v = 2s$, $b_2 = s(2s-1)$, $r_2 = 3(2s-1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 3,4,...$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.4) with $b_1 = 1$ the variance matrix of errors $\sigma^2 g\mathbf{I}_{b_2+1}$.

Proof. The main idea of the proof is to check that parameters given above satisfy the condition (2.3).

Theorem 2.14. Let $0 < \rho < 1$ and $k_{12} \neq k_{22}$. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (1.1), is regular D-optimal if and only if (2.3) holds and

$$b_1 - 2r_1 + \frac{\lambda_{12}(v-1)(k_{12}-k_{22})}{2k_{12}k_{22}} = 0. \tag{2.5}$$

Proof. For the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1) and from the condition $\mathbf{X}^{'}\mathbf{X} = m\mathbf{I}_p$ of Theorem 2.1 we obtain (2.3). The equality $\mathbf{X}_1^{'}\mathbf{1}_n = \mathbf{0}_p$ implies that in each column of the matrix $\mathbf{X}$ the number of elements equal to –1 is the same as the number of elements equal to 1. Thus $b_1 - r_1 + r_{12} = r_1 + r_{22}$. Since for $k_{12} \neq k_{22}$, we have $r_{12} = \frac{\lambda_{12}(v-1)}{2k_{22}}$, $r_{22} = \frac{\lambda_{12}(v-1)}{2k_{12}}$ then we obtain (2.5).

Theorem 2.15. Let $0 < \rho < 1$. The existence of the balanced incomplete block designs and the balanced bipartite weighing design with the parameters

(i) $v = 8$, $b_1 = 28$, $r_1 = 7$, $k_1 = 2$, $\lambda_1 = 1$ and $v = 8$, $b_2 = 56$, $r_2 = 56$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 30$, $\lambda_{22} = 26$,

(ii) $v = 9$, $b_1 = 24s$, $r_1 = 8s$, $k_1 = 3$, $\lambda_1 = 2s$ and $v = 9$, $b_2 = 36$, $r_2 = 16$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 3$, $s = 1,2$,

(iii) $v = 12$, $b_1 = 11s$, $r_1 = 11$, $k_1 = 12s^{-1}$, $\lambda_1 = 12s^{-1} - 1$ and

(a) for $s = 3$, $b_2 = 66$, $r_2 = 33$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$,

(b) for $s = 4$, $b_2 = 132$, $r_2 = 132$, $k_{12} = 5$, $k_{22} = 7$, $\lambda_{12} = 70$, $\lambda_{22} = 62$,

(iv) $v = 13$, $b_1 = 39$, $r_1 = 15$, $k_1 = 5$, $\lambda_1 = 5$ and $v = 13$, $b_2 = 39$, $r_2 = 15$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$,

(v) $v = 15$, $b_1 = 42$, $r_1 = 14$, $k_1 = 5$, $\lambda_1 = 4$ and $v = 15$, $b_2 = 105$, $r_2 = 56$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 15$, $\lambda_{22} = 3$,

(vi) $v = 16$, $b_1 = 40$, $r_1 = 15$, $k_1 = 6$, $\lambda_1 = 5$ and $v = 16$, $b_2 = 80$, $r_2 = 20$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 2$, $\lambda_{22} = 2$,

(vii) $v = 25$, $b_1 = 40$, $r_1 = 16$, $k_1 = 10$, $\lambda_1 = 6$ and $v = 25$, $b_2 = 100$, $r_2 = 16$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 1$, $\lambda_{22} = 1$,

implies the existence of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(-1,0,1)$ in the form (2.4) with the variance matrix of errors $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1.1).

Proof. To see this Theorem we observe that parameters (i)-(vii) satisfy the conditions (2.3) and (2.5).

3. Discussion

It is worth to point out, that it is not possible to determine D-optimal chemical balance weighing design in any class $\mathbf{\Phi}_{n\times p}(-1,0,1)$. Therefore, in present paper we wide the class of D-optimal designs given in literature, see Ceranka and Graczyk (2014-2017). The methods of construction of the regular D-optimal chemical balance weighing designs with non-negative correlated errors based on the set of the incidence matrices of the balanced bipartite weighing designs and the ternary balanced block design were given in Ceranka and Graczyk (2016).

Let $\rho = 0$ and let $p = 10$. Based on the Theorem 2.3 (i) of Ceranka and Graczyk (2016) we are able to determine the regular D-optimal chemical balance weighing design in the class $\mathbf{\Phi}_{119\times 10}(-1,0,1)$, whereas based on the Theorem 2.8 given above in the class $\mathbf{\Phi}_{40\times 10}(-1,0,1)$. Moreover, based on the results given in Ceranka and Graczyk (2016) we are not able to determine the regular D-optimal chemical balance weighing design for $p = 8$ objects under assumptions $\rho = 0$ and $0 < \rho < 1$, but such design is indicated in the class $\mathbf{\Phi}_{36\times 8}(-1,0,1)$ (Theorem 2.7) for $\rho = 0$ and in the class $\mathbf{\Phi}_{84\times 8}(-1,0,1)$ for $0 < \rho < 1$. If $0 < \rho < 1$, then the regular D-optimal design we determine in the class $\mathbf{\Phi}_{140\times 12}(-1,0,1)$ (Th. 2.6 (ii) $s = 3$) in Ceranka and Graczyk (2016) and here in the class $\mathbf{\Phi}_{99\times 12}(-1,0,1)$. It seems to be important from the point of view of experimental costs, as in present paper we are able to indicate the regular D-optimal design in smallest number of measurements.

4. Example

Let us consider the experiment in that we determine unknown measurements of $p = 6$ objects by used of $n = 16$ measurements assuming that the correlation between measurement errors equals $\rho = 0$. We construct the matrix $\mathbf{X} \in \mathbf{\Phi}_{16\times 6}(-1,0,1)$ according to the Theorem 2.5(i). Let $\mathbf{N}_1$ be the incidence matrix of the balanced incomplete block design with the parameters $v = 6$, $b_1 = 10$, $r_1 = 5$, $k_1 = 3$, $\lambda_1 = 2$ and let $\mathbf{N}_2^*$ be the incidence matrix of the balanced bipartite weighing design with the parameters $v = 6$, $b_2 = 6$, $r_2 = 6$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 2$, $\lambda_{22} = 4$ given in the forms

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \mathbf{N}_2^* = \begin{bmatrix} 1_1 & 1_2 & 1_2 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_1 & 1_2 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_2 & 1_1 & 1_2 & 1_2 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_1 & 1_2 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_2 & 1_1 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_2 & 1_2 & 1_1 \end{bmatrix},$$

where 1_ς denotes the element belonging to the ς^{th} subblock, respectively, $\varsigma = 1,2$.

According to the formula (2.1) we form the matrix $\mathbf{X} \in \mathbf{\Phi}_{16\times 6}(-1,0,1)$ of the regular D-optimal chemical balance weighing design as

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{bmatrix}$$

5. Conclusions

The problem of the determining D-optimal chemical balance weighing design in any class $\mathbf{\Phi}_{n\times p}(-1,0,1)$ is still open. From the theoretical point of view, the conditions under that the design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ is regular D-optimal under assumption the errors are equal correlated and they have the same variances are given above. However, the construction methods of the matrices satisfying these conditions for any number of objects and measurements are not given. The construction problems require further studies.

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