Colloquium Biometricum 39
2009, 53–62

# ON SOME METHOD OF CONSTRUCTION OPTIMUM BIASED SPRING BALANCE WEIGHING DESIGN

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**Summary**

The paper deals with certain biased spring balance weighing designs. The method of construction is given. The incidence matrices of two group divisible designs with the same association scheme have been used to construct these designs. Theoretical research is illustrated by example of using this method to the analyzing the lines of legume.

**Key words and phrases:** group divisible design, optimum biased spring balance weighing design

**Classification AMS 2000**: 62K15

## 1. Introduction

The formulation to a standard weighing design problem calls for some objects with unknown weights and a weight measuring device which is popularly known as a balance. In a spring balance, there is only one pan and any number of objects can be placed on the pan. Then the pointer provides a rearing which represents the total weight of the objects on the pan. If we want to determine any measurements of given number $p$ of objects it seams to be the best way to measure each object separately. Assuming in $n$ measurements there are not systematic errors and the variance of the error is $\sigma^2$, then the variance of each of the estimators of unknown measurements of objects is equal to $2\sigma^2$. If we will take these objects to measurements in some combinations then we reduce the variance to $\sigma^2/2$ (See Yates (1933), Banerjee (1950, 1975)). Nowadays, spring balance weighing design there is the name for experimental design connected not only with balance, but with each experiment which result we can describe as linear combination of unknown measurements of objects with factors of this combination equal to 1 or 0. A study of weighing designs is supposed to be helpful in routine weighing operations to determine weights of light objects. Moreover, "the design are used to a great variety of problems of measurements, not only for weights, but of lengths, voltages and resistances, concentrations of chemicals in mixtures, in fact any measurements such that the measure of combination is known linear function of the separate measures with numerically equal coefficient" (Mood (1946)). The illustration of the application of the theory of weighing designs was given, for example, by Banerjee (1950). Various aspects of spring balance weighing designs have been studied by Raghavarao (1971), Banerjee (1975), Federer at all (1976).

Suppose, there are $p$ objects of true unknown measurements $w_1$, $w_2$,..., $w_p$, respectively, and we wish to estimate them employing $n$ measurement operations using a spring balance. Let $y_1$, $y_2$,..., $y_n$, denote respectively recorded observations in these $n$ operations. It is assumed that the observations follow the standard regression model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}, \tag{1.1}$$

where $\mathbf{X} = (x_{ij})$ is called the design matrix, $i$ = 1,2,...,$n$, $j$ = 1,2,..., $p$, and has the elements 1 or 0 according whether in the $i$ th weighing operation the $j$ th object is placed on the pan or not, $\mathbf{e}$ is $n\times 1$ random vector of errors. We assume, in the model (1.1) errors have the same variances and they are uncorrelated, i.e. $E(\mathbf{e}) = \mathbf{0}_n$ and $E(\mathbf{ee}') = \sigma^2\mathbf{I}_n$, where $\mathbf{0}_n$ is an $n\times 1$ null vector, $\mathbf{I}_n$ is $n\times n$ identity matrix. The normal equations for estimating $\mathbf{w}$ are of the form $\mathbf{X}'\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{y}$, where $\hat{\mathbf{w}}$ is the column vector of the weights estimated by the least squares method. A spring balance weighing design is said to be singular or not singular depending on whether the matrix $\mathbf{X}'\mathbf{X}$ is singular or not singular, respectively. If $\mathbf{X}$ is of full column rank the least squares estimates of $\mathbf{w}$ are given by $\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$ and the covariance matrix of $\hat{\mathbf{w}}$ is $\mathrm{V}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}$.

In metrology, dynamical systems theory, computational mechanics or statistics appears bias as a term used to describe a tendency or preference towards a particular result or as a bias of a measurement system or estimate method, which leads to systematic errors, namely produces the readings or results which are consistently too high or too low, relative to a given actual value of the measured or estimated value. If the balance is not corrected for bias, we can assume one of the true weights, say the first, to be representing the bias, and we choose the first column of $\mathbf{X}$ to be vector of ones. In any nonsingular spring balance weighing design the optimality condition is never satisfied. Particularly, if in nonsingular spring balance design given by the matrix **X** first column is column of ones then $\mathrm{V}(\hat{w}_1) = p\sigma^2/n,\ \ \mathrm{V}(\hat{w}_j) \geq 4\sigma^2/n,\ j = 2,3,...,p$ (See Moriguti (1954)). Hence

$$\mathbf{X} = \begin{bmatrix}\mathbf{1}_n & \mathbf{X}_1\end{bmatrix}, \tag{1.2}$$

where $\mathbf{X}_1$ as a part of the design matrix $\mathbf{X}$ is $n \times (p-1)$ matrix with elements equal to 1 or 0, $\mathbf{1}_n$ is $n \times 1$ vector of units.

**Definition 1.1.** The design with $\mathbf{X}$ in the form (1.2) is called biased spring balance weighing design (BSBWD).

**Definition 1.2.** Any nonsingular BSBWD is said to be optimal if it estimates each of the weights with minimum variance, i.e.

$$\mathrm{Var}(\hat{w}_j) = \frac{4\sigma^2}{n}, \quad j = 2,3,...,p.$$

The construction method of a BSBWD based on the incidence matrices of the balanced incomplete block designs and the conditions under which the design is optimal are presented in Ceranka and Katulska (1987b). Moreover, it is shown how this theory may be utilized to obtain treatment and experiment designs to estimate differences in legume content between pair of lines in an experiment oversees with grass species. Application of the theory of optimum BSBWD to the analysis of experiments with mixtures of cultivars is considered in Ceranka and Katulska (1987a,c). The problems connected with estimation of the individual weights of objects in BSBWD are considered in Katulska (1989). From this paper we have the following Theorems.

**Theorem 1.1.** For a nonsingular BSBWD with $\mathbf{X}$ given by (1.2) the variances of each of the estimated weights are minimal if and only if

$$\mathbf{X}_1^{'}\mathbf{X}_1 = \frac{n}{4}\left(\mathbf{I}_{p-1} + \mathbf{1}_{p-1}\mathbf{1}_{p-1}^{'}\right). \tag{1.3}$$

Now, let $\mathbf{X}$ be partitioned as

$$\mathbf{X} = \begin{bmatrix} 1 & \mathbf{x}^{'} \\ \mathbf{1}_{n-1} & \mathbf{X}_2 \end{bmatrix} = \left[\mathbf{1}_n \quad \mathbf{X}_1\right], \tag{1.4}$$

where $\mathbf{X}_2$ as a part of the design matrix $\mathbf{X}$ is $(n-1)\times(p-1)$ matrix with elements equal to 1 or 0.

**Theorem 1.2.** Let $\mathbf{X}_2^{'}\mathbf{X}_2 = \frac{n}{4}\mathbf{I}_p + h\mathbf{1}_p\mathbf{1}_p^{'}$. The design matrix **X** given by (1.4) is the design matrix of an optimum BSBWD if and only if $\mathbf{x} = q\mathbf{1}_{p-1}$ and $h = \frac{n}{4} - q$ where $q$ = 0 or 1.

In the present paper we study the method of construction the design matrix $\mathbf{X}$ given by (1.2) or (1.4) for an optimum BSBWD based on Theorems 1.1 or 1.2, respectively, using the incidence matrices of two group divisible designs with the same association scheme.

## 2. Optimum weighing design

Now, we present the definitions of the partially balanced block design with two associate classes and the group divisible design given by Raghavarao (1971).

A partially balanced block design with two associate classes (PBBDTAC) there is an arrangement of $v$ treatments in $b$ blocks, each of size $k$, in such a way, that each treatment occurs at most ones in each block, occurs in exactly $r$ blocks. There can be established a relation of association between any two treatments satisfying the following requirements: two treatments are either first associates or second associates, each treatment has exactly $n_\varsigma$ $\varsigma$ th associates, $\varsigma = 1,2$. Giving any two treatments which are $\varsigma$ th associates, the number of treatment common to the $\xi$ th associate of the first and the $\eta$ th associate of the second is $p_{\xi\eta}^{\varsigma}$ and its independent of the pair of treatments we start with. Also $p_{\eta\xi}^{\varsigma} = p_{\xi\eta}^{\varsigma}$, $\varsigma, \xi, \eta = 1,2$. Two treatments which are $\varsigma$ th associate occur together in exactly $\lambda_{\varsigma}$ blocks, $\varsigma = 1,2$. The numbers $v, b, r, k, n_1, n_2, \lambda_1, \lambda_2$ are called the parameters of the first kind, whereas the numbers $p_{\xi\eta}^{\varsigma}$ are called the parameters of the second kind.

A group divisible design (GDD) is a PBBDTAC for which the $v$ treatments may be divided into $m$ groups of $q$ distinct treatments each, such that treatments belonging to the same group are first associates and two treatments belonging to the different groups are second associates $v = mq,\ n_1 = q-1,\ n_2 = q(m-1),\ (q-1)\lambda_1 + (m-1)\lambda_2 = r(k-1)$.

Consider GDD with parameters $v, b_h, r_h, k_h, \lambda_{1h}, \lambda_{2h}$, $h = 1,2$ with the same association scheme (See Raghavarao (1971), Clatworthy (1973)). Let $\mathbf{N}_h$ denote the incidence matrix of the order $v \times b_h$, $h = 1,2$. Additionally we assume that $\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = \lambda$. Then

$$\mathbf{N}_1\mathbf{N}_1^{'} + \mathbf{N}_2^{'}\mathbf{N}_2 = (r_1 + r_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v^{'}. \tag{2.1}$$

Consider the first design matrix $\mathbf{X}$ given by (1.2), where $\mathbf{X}_1 = \begin{bmatrix} \mathbf{N}_1^{'} \\ \mathbf{N}_2^{'} \end{bmatrix}$. Then

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}_{b_1} & \mathbf{N}_1^{'} \\ \mathbf{1}_{b_2} & \mathbf{N}_2^{'} \end{bmatrix} \tag{2.2}$$

is the design matrix of a BSBWD. In this design $p = v + 1$ and $n = b_1 + b_2$. The following corollary is the consequence of Theorem 2.3 given by Katulska (1989).

**Corollary 2.1.** BSBWD with matrix $\mathbf{X}$ given by (2.2) is optimal if and only if BSBWD with the matrix

$$\mathbf{X} = \begin{bmatrix} \mathbf{1}_{b_1} & \mathbf{1}_{b_1}\mathbf{1}_v^{'} - \mathbf{N}_1^{'} \\ \mathbf{1}_{b_2} & \mathbf{1}_{b_2}\mathbf{1}_v^{'} - \mathbf{N}_2^{'} \end{bmatrix} \tag{2.3}$$

is optimal.

It may be noted that the designs (2.2) and (2.3) are related by using the complementary GDD.

From (1.3) and (2.1) we have the following corollary.

**Corollary 2.2.** BSBWD with the matrix $\mathbf{X}$ given by (2.2) is optimal if and only if

$$b_1 + b_2 = 4(r_1 + r_2 - \lambda) \tag{2.4}$$

and

$$r_1 + r_2 = 2\lambda . \tag{2.5}$$

Now, consider the matrix $\mathbf{X}$ given by (1.4) where $\mathbf{x} = \mathbf{1}_v$ and $\mathbf{X}_2 = \begin{bmatrix} \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix}$. Then

$$\mathbf{X} = \begin{bmatrix} 1 & \mathbf{1}_v' \\ \mathbf{1}_{b_1} & \mathbf{N}_1' \\ \mathbf{1}_{b_2} & \mathbf{N}_2' \end{bmatrix} \tag{2.6}$$

is the design matrix of BSBWD. In this design $p = v + 1$ and $n = b_1 + b_2 + 1$. The following corollary is the consequence of the (1.3) and Theorem 1.2.

**Corollary 2.3.** BSBWD with matrix $\mathbf{X}$ given by (2.6) is optimal if and only if

$$b_1 + b_2 + 1 = 4(r_1 + r_2 - \lambda) \tag{2.7}$$

and

$$r_1 + r_2 = 2\lambda + 1 . \tag{2.8}$$

From Corollary 2.1, relation (1.3) and Theorem 1.2 we have the following corollary.

**Corollary 2.4.** BSBWD with the matrix $\mathbf{X}$ equals to

$$\mathbf{X} = \begin{bmatrix} 1 & \mathbf{0}_v^{'} \\ \mathbf{1}_{b_1} & \mathbf{1}_{b_1}\mathbf{1}_v^{'} - \mathbf{N}_1^{'} \\ \mathbf{1}_{b_2} & \mathbf{1}_{b_2}\mathbf{1}_v^{'} - \mathbf{N}_2^{'} \end{bmatrix} \qquad (2.9)$$

is optimal if and only if (2.7) and (2.8) are satisfied.

## 3. Group divisible design leading to optimum weighing designs

Corollary 2.2 shows that if the parameters of two GDD's with the same association scheme satisfy (2.4), (2.5) and the condition $\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = \lambda$ is fulfilled then a BSBWD with the design matrix $\mathbf{X}$ given by (2.2) is optimal. Based on Clatworthy (1973) the list of all parameters combinations of the existing GDD's is given in the Table 1.

Let note, the additionally numbers given in the first column of the Tables denote the numbers of designs given in Clatworthy (1973).

**Table 1.** Parameters of the group divisible designs

| First design | | | | | | | Second design | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Type of GDD | $v$ | $b$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ | Type of GDD | $v$ | $b$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ |
| S18 | 8 | 4 | 3 | 6 | 3 | 2 | R54 | 8 | 8 | 3 | 3 | 0 | 1 |
| S19 | 8 | 8 | 6 | 6 | 6 | 4 | R25 | 8 | 16 | 6 | 3 | 0 | 2 |
| S20 | 8 | 12 | 9 | 6 | 9 | 6 | R57 | 8 | 24 | 9 | 3 | 0 | 3 |
| R58 | 8 | 24 | 9 | 3 | 2 | 3 | R164 | 8 | 12 | 9 | 6 | 7 | 6 |
| S82 | 12 | 4 | 3 | 9 | 3 | 2 | R145 | 12 | 12 | 5 | 5 | 1 | 2 |
| S83 | 12 | 8 | 6 | 9 | 6 | 4 | R148 | 12 | 24 | 10 | 5 | 2 | 4 |

Moreover, Corollary 2.3 says that if the parameters of two GDD's satisfy the conditions (2.7) and (2.8) then BSBWD with the matrix $\mathbf{X}$ given by (2.6) is optimal. The list of all parameters combinations of the existing GDD's is given in the Table 2.

**Table 2.** Parameters of the group divisible designs

| First design | | | | | | | Second design | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Type of GDD | $v$ | $b$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ | Type of GDD | $v$ | $b$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ |
| SR6 | 6 | 9 | 3 | 2 | 0 | 1 | R94 | 6 | 6 | 4 | 4 | 3 | 2 |
| S32 | 14 | 7 | 3 | 6 | 3 | 1 | SR80 | 14 | 8 | 4 | 7 | 0 | 2 |

In the Table 3 there are presented parameters of GDD's given by Clatworthy (1973). Based on these parameters we are able to construct the optimum BSBWD given by design matrix $\mathbf{X}$ of the form (2.9).

**Table 3.** Parameters of the group divisible designs

| First design | | | | | | | Second design | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Type of GDD | $v$ | $b$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ | Type of GDD | $v$ | $b$ | $r$ | $k$ | $\lambda_1$ | $\lambda_2$ |
| S1 | 6 | 3 | 2 | 4 | 2 | 1 | SR18 | 6 | 4 | 2 | 3 | 0 | 1 |
| S58 | 12 | 15 | 10 | 8 | 10 | 6 | R147 | 12 | 24 | 10 | 5 | 0 | 4 |
| S59 | 14 | 7 | 4 | 8 | 4 | 2 | SR80 | 14 | 8 | 4 | 7 | 0 | 2 |

## 4. Example

In this chapter we study methods of the construction of the optimum BSBWD in view of employing the theory of this design to the analysis of experiments with mixtures. Let us consider the experiment in which we compare $v = 6$ lines of a legume, namely A, B, C, D, E, F, over seeded with grass species. It is assumed that more than one legume line can be grown on each experimental plot and that the competitive effect between lines is negligible. Then the yield consists of the weight of grass, the weight of weeds and the weight of 6 legume lines. The experiment corresponds to a BSBWD in which the grass is identified with the bias and the legume lines are identified as objects. The individual lines are not treatments of the experiment, but any treatment is a composite of a combination of any number of lines from A – F. The design matrix **X** f an optimum BSBWD we construct from the incidence matrices of GDD with parameters $v = 6, b_1 = 9, r_1 = 3, k_1 = 2, \lambda_{11} = 0, \lambda_{21} = 1$ (SR6 from Clatworthy (1973)) and GDD with parameters $v = 6, b_2 = 6, r_2 = 4, k_2 = 4, \lambda_{12} = 3, \lambda_{22} = 2$ (R94 from Clatworthy (1973)), where

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{N}_2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

From the above consideration and Corollary 2.3 it follows that $\mathbf{X}$ given by (2.6) is the matrix of an optimum BSBWD in the form

$$\mathbf{X}' = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Hence we have the following treatment designs: treatment 1: grass and lines A, B, C, D, E, F, treatment 2: grass and lines A, B, treatment 3: grass and lines C, D, and so on. Each of unknown measure of object is estimated with the minimal variance equal $\sigma^2/4$. For more details see Federer and all (1976).

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# O PEWNEJ METODZIE KONSTRUKCJI OPTYMALNYCH SPRĘŻYNOWYCH UKŁADÓW WAGOWYCH

## Streszczenie

W pracy przedstawiona została metoda konstrukcji optymalnego sprężynowego układu wagowego w oparciu o macierze incydencji dwóch układów o grupach podzielnych o tym samym schemacie partnerstwa. Rozważania teoretyczne zostały uzupełnione przykładem dotyczącym roślin strączkowych.

**Słowa kluczowe:** optymalny sprężynowy układ wagowy, układ o grupach podzielnych

**Klasyfikacja AMS 2000**: 62K15