Colloquium Biometricum 39
2009, 63–68

CONSTRUCTION OF EXPERIMENTAL DESIGNS BY COMBINING SOME GD DESIGNS

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Summary

Balanced incomplete block designs (BIB) are used in many constructions. These designs, next to many merits, possess also some negative sides: they do not exist for many parameters, or they have too many blocks and experimental units. This paper presents situations where BIB designs can be replaced by group divisible block designs (GD).

Key words and phrases: balanced incomplete block designs, group divisible block designs, Kronecker product of matrices, split-plot designs

Classification AMS 2000: 62K10, 62K15

1. Introduction

In BIB designs, each basic contrast is estimable with the same variance $\frac{\sigma^2}{\mu}$, where μ is the only non-zero eigenvalue of the information matrix **C**. In

GD designs, basic contrasts are estimable with two variances: $\frac{\sigma^2}{\mu_1}$ oraz $\frac{\sigma^2}{\mu_2}$, where μ_1 and μ_2 are non-zero eigenvalues of **C**. The smaller is the difference between $\frac{\sigma^2}{\mu_1}$ and $\frac{\sigma^2}{\mu_2}$, the more "similar" is the GD design to the BIB design. In this paper, we replace the BIB designs with such GD designs for which

$$\left|\frac{1}{\mu_1}-\frac{1}{\mu_2}\right| \le 0.05, \tag{1.1}$$

it is when the variances $\frac{\sigma^2}{\mu_1}$ and $\frac{\sigma^2}{\mu_2}$ differ at the most by $0.05\sigma^2$.

2. Results

2.1 GD design

A GD design is a block design based on $v = mn$ treatments (being arranged into m groups of n treatments each), consisting of b blocks of size k ($k<v$), such that each treatment occurs in r blocks. In the GD design, two treatments belonging to the same group that occur together in λ_1 blocks are called first associates, while those belonging to different groups that occur together in λ_2 blocks are called second associates. If n_i denotes the number of treatments which are ith associates of any treatment, then $n_1 = n-1$ and $n_2 = (m-1)n$. The numbers v, b, r, k, λ_1, λ_2, m, n are called the parameters of the design.

In case of $\lambda_1 = \lambda_2 = \lambda$ we obtain balanced incomplete block (BIB) design.

If $\mathbf{N} = (n_{ij})$ is a $(v\times b)$ incidence matrix of GD design, then from the above definition we have

$$\mathbf{NN'} = r\mathbf{A}_0 + \lambda_1\mathbf{A}_1 + \lambda_2\mathbf{A}_2, \tag{2.1}$$

where $\mathbf{A}_0 = \mathbf{I}_v$, $\mathbf{A}_1 = \mathbf{I}_m \otimes (\mathbf{J}_n - \mathbf{I}_n)$, $\mathbf{A}_2 = (\mathbf{J}_m - \mathbf{I}_m)\otimes \mathbf{J}_n$, while $\mathbf{I}_x$ is the unit matrix of order x, $\mathbf{J}_x$ is a $(x\times x)$-matrix of ones and $\otimes$ denotes the

Kronecker product of matrices. Note that $\mathbf{A}_1\mathbf{1}_v = n_1\mathbf{1}_v$, $\mathbf{A}_2\mathbf{1}_v = n_2\mathbf{1}_v$, where $\mathbf{1}_v$ denotes vector of *v* ones.

It is known that the information matrix of GD has the form:

$$\mathbf{C} = r\mathbf{I}_v - \frac{1}{k}\mathbf{NN}',$$

where $\mu_0 = 0$, $\mu_1 = \frac{r(k-1)+\lambda_1}{k}$ and $\mu_2 = \frac{v\lambda_2}{k}$ are eigenvalues of **C** with multiplicities $\alpha_0 = 1$, $\alpha_1 = m(n-1)$, $\alpha_2 = m-1$, respectively.

GD designs have favourable statistical properties connected with the estimability of the basic and elementary treatment contrasts. Consider the vectors:

$\mathbf{p}_{1j} = \mathbf{1}_m \otimes \mathbf{p}_j^n / \left\|\mathbf{1}_m \otimes \mathbf{p}_j^n\right\|, \ j = 1,\dots,n-1,$

$\mathbf{p}_{1ij} = \mathbf{p}_i^m \otimes \mathbf{p}_j^n / \left\|\mathbf{p}_i^m \otimes \mathbf{p}_j^n\right\|,$

$\mathbf{p}_{2i} = \mathbf{p}_i^m \otimes \mathbf{1}_n / \left\|\mathbf{p}_i^m \otimes \mathbf{1}_n\right\|, \ i = 1,\dots,m-1,$

where $\mathbf{p}_i^m$ are any collections of *m*–1 mutually orthogonal vectors with *m* components, satisfying the condition $(\mathbf{p}_i^m)'\mathbf{1}_m = 0$. Similarly, the vectors $\mathbf{p}_j^n$ are any collections of *n*–1 vectors with *n* components satisfying $(\mathbf{p}_j^n)'\mathbf{1}_n = 0$, and $\|\mathbf{x}\| = \sqrt{\mathbf{x}'\mathbf{x}}$. For example we can assume:

$\mathbf{p}_1^m = [m-1,-1,\dots,-1]'$, $\mathbf{p}_2^m = [0,m-2,-1,\dots,-1]',\dots,$ $\mathbf{p}_{m-1}^m = [0,..,0,1,-1]'$
and
$\mathbf{p}_1^n = [n-1,-1,\dots,-1]'$, $\mathbf{p}_2^n = [0,n-2,-1,\dots,-1]',\dots,$ $\mathbf{p}_{n-1}^n = [0,..,0,1,-1]'$.

It is known (see Brzeskwiniewicz, 1995) that in fixed linear models for block design variances of the estimator of treatment contrasts $\mathbf{p}_{1j}'\mathbf{t}$, $\mathbf{p}_{2i}'\mathbf{t}$ and $\mathbf{p}_{1ij}'\mathbf{t}$, when $\mu_1 \neq 0$ and $\mu_2 \neq 0$, are equal to

$$Var(\widehat{\mathbf{p}_{1j}'\mathbf{t}}) = \frac{\sigma^2}{\mu_1},\ Var(\widehat{\mathbf{p}_{1ij}'\mathbf{t}}) = \frac{\sigma^2}{\mu_1}, Var(\widehat{\mathbf{p}_{2i}'\mathbf{t}}) = \frac{\sigma^2}{\mu_2}, \qquad (2.2)$$

where **t** is an unknown column vector of treatment parameters and σ^2 denotes the error variance of the intra-block analysis.

2.2. BIB designs

A balanced incomplete block design (see e.g. Raghavarao, 1971) is an arrangement of v treatments in b blocks of sizes k such that every treatment occurs r times and every pair of distinct treatments is contained in exactly λ blocks. The numbers v, b, r, k, λ are called the parameters of the design, $\mu_0 = 0$ and $\mu = \frac{r(k-1)+\lambda}{k}$ are eigenvalues of **C** with multiplicities $\alpha_0 = 1$ and $\alpha_1 = v-1$, respectively.

Table 1. The GD designs from catalogue of Clatworthy (1973) with $\left|\frac{1}{\mu_1} - \frac{1}{\mu_2}\right| \leq 0.05$

reference	number
S	4, 5, 18-20, 24-26, 28, 31, 33, 34, 39, 41, 49-52, 56-60, 63, 65, 67, 72, 74, 76, 80-87, 89, 90, 92, 93, 96-99, 103-105, 109-111, 118, 119, 121, 124
SR	22, 29, 34, 35, 38-40, 43, 45, 47-51, 53-57, 59-110
R	6, 10, 16, 17, 27, 30, 36, 37, 40, 43, 46, 48, 50-52, 57, 58, 62, 65, 67, 68, 70, 71, 75, 77-79, 81-83, 87-89, 90-93, 95-101, 103, 106, 108-113, 116-118, 122, 125-129, 130-132, 134, 136, 138-142, 144, 145, 147, 148, 150, 151, 153-155, 159-165, 169-172, 174-177, 179, 180, 181, 183-185, 189-194, 196, 197, 200-205, 209

Table 2. The GD designs from Table 1 with $b < v$.

No.	v	r	k	b	m	n	λ_1	λ_2	No.	v	r	k	b	m	n	λ_1	λ_2
S18	8	3	6	4	4	2	3	2	SR66	12	4	6	8	6	2	0	2
S28	12	5	6	10	6	2	5	2	SR75	30	5	6	25	6	5	0	1
S51	10	4	8	5	5	2	4	3	SR80	14	4	7	8	7	2	0	2
S59	14	4	8	7	7	2	4	2	SR81	14	6	7	12	7	2	0	3
S63	16	7	8	14	8	2	7	3	SR84	21	6	7	18	7	3	0	2
S67	20	6	8	15	10	2	6	2	SR90	12	6	8	9	4	3	3	4
S74	32	5	8	20	16	2	5	1	SR91	16	6	8	12	8	2	0	3
S82	12	3	9	4	4	3	3	2	SR96	56	7	8	49	8	7	0	1
S83	12	6	9	8	4	3	6	4	SR99	18	6	9	12	9	2	0	3
S85	15	6	9	10	5	3	6	3	SR100	18	8	9	16	9	2	0	4
S86	18	5	9	10	6	3	5	2	SR103	36	8	9	32	9	4	0	2
S89	21	6	9	14	7	3	6	2	SR104	72	8	9	64	9	8	0	1
S92	27	8	9	24	9	3	8	2	SR106	20	6	10	12	10	2	0	3
S98	12	5	10	6	6	2	5	4	SR107	20	8	10	16	10	2	0	4
S103	15	8	10	12	3	5	8	4	SR109	30	9	10	27	10	3	0	3
S109	20	9	10	18	10	2	9	4	SR110	90	9	10	81	10	9	0	1
S110	22	5	10	11	11	2	5	2									
S118	42	5	10	21	21	2	5	1									
S121	50	6	10	30	25	2	6	1									

Ceranka and Goszczurna (1994) give a complete list of these incidence matrices for v<20, $r \le 15$, $2 \le k \le v/2$ and an additional remark about the construction with $v/2 < k < v-1$.

In Table 1 there are symbols of GD designs from catalogue of Clatworthy (1973) with (1.1). In Table 2 there are designs from Table 1 with $b < v$ (for BIB design, we have $b \ge v$).

3. Example

Let us consider a two-factor experiment of split-plot type in which a factor A occurs on 10 levels: A_1, …, A_{10} and the second factor B occurs on 12 levels: B_1, …, B_{12}. We consider two BIB designs from catalogue of Ceranka and Goszczurna (1994) with parameters v = 10, r = 6, k = 4, b = 15, λ=2 with incidence matrix $\mathbf{N}_A$ and v = 12, r = 11, k = 6, b = 22, λ=3 with incidence matrix $\mathbf{N}_B$, respectively. Matrix $\mathbf{N}_1 = \mathbf{N}_A \otimes \mathbf{N}_B$ is the incidence matrix for the subplots in split-plot design and it has $10 \cdot 12 = 120$ rows, $15 \cdot 22 = 330$ columns and 7920 experimental units.

We consider two GD designs from Table 2: S51 and S82. Matrix $\mathbf{N}_1$ has $10 \cdot 12 = 120$ rows, $5 \cdot 4 = 20$ columns and 1440 experimental units, what is very favourable situation.

References

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KONSTRUKCJA EKSPERYMENTALNYCH UKŁADÓW Z WYKORZYSTANIEM PEWNYCH UKŁADÓW GD

Streszczenie

W wielu konstrukcjach wykorzystuje się układy zrównoważone o blokach niekompletnych (BIB). Układy te oprócz wielu zalet, mają wady: nie istnieją dla wielu parametrów albo posiadają za dużo bloków i jednostek doświadczalnych. W tej pracy podano sytuacje, w których układy BIB mogą zostać zastąpione układami blokowymi z grupowo podzielnymi obiektami (GD).

Słowa kluczowe: zrównoważone układy o blokach niekompletnych, układy blokowe z grupowo podzielnymi obiektami, iloczyn Kroneckera macierzy

Klasyfikacja AMS 2000: 62K10