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SOME CONSTRUCTION OF REGULAR E–OPTIMAL SPRING BALANCE WEIGHING DESIGNS FOR EVEN NUMBER OF OBJECTS

Bronisław Ceranka, Małgorzata Graczyk

Department of Mathematical and Statistical Methods
Poznań University of Life Sciences
Wojska Polskiego 28, 60–637 Poznań, Poland
bronicer@up.poznan.pl; magra@up.poznan.pl

Summary

In the paper, the problem of the construction of the E–optimal spring balance weighing design is discussed. The incidence matrices of the partially incomplete block designs with two associate classes are used to that construction.

Key words and phrases: E–optimal design, partially incomplete block design, spring balance weighing design

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1. Introduction

In metrology, dynamical system theory, computational mechanics and statistics spring balance weighing designs are discussed. The statistical problem is to estimate the vector of unknown parameters when the observations undergo model $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$, where $\mathbf{y}$ is an $b \times 1$ random vector of the observations, $\mathbf{w} = (w_1, w_2, ..., w_v)'$ is a vector representing unknown measurements of objects, $\mathbf{X} = (x_{ij})$, $i = 1,2,...,b$, $j = 1,2,...,v$, $\mathbf{X} \in \mathbf{\Phi}_{b \times v}(0,1)$, where $\mathbf{\Phi}_{b \times v}(0,1)$

denotes the class of $b \times v$ matrices having entries $x_{ij} = 0$ or 1, $\mathbf{e}$ is an $b \times 1$ random vector of errors. We assume that errors are uncorrelated and have constant variance σ^2, i.e. $\mathrm{E}(\mathbf{e}) = \mathbf{0}_b$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2 \mathbf{I}_b$, where $\mathbf{0}_b$ is $b \times 1$ vector of zeros, $\mathbf{I}_b$ is the $b \times b$ identity matrix. If the design matrix $\mathbf{X}$ is of full column rank, then all w_j are estimable and the variance matrix of their best linear unbiased estimator is $\sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1}$. The matrix $\left(\mathbf{X}'\mathbf{X}\right)^{-1}$ is called the information matrix of $\mathbf{X}$.

The optimality problem is concerned with efficient estimation in some sense by a proper choice of the design matrix $\mathbf{X}$ among many designs at our disposal in the class $\mathbf{\Phi}_{b \times v}(0,1)$. In many problems concerning experimental design, the criterion of E–optimality is used. E–optimal design, is such a design where the maximal eigenvalue of the information matrix is minimal. The concept of E–optimality was considered, for instance, in the book of Pukelsheim (1993). For experimental designs criterion of E–optimality is interpreted as minimizing the maximum variance of the component estimates of the parameters.

The purpose of this paper is to present new construction method, which gives the E–optimal spring balance weighing designs.

2. The design matrix of the spring balance weighing design

Let us recall the theorem

Theorem 2.1. (Jacroux and Notz (1983)) For even v, any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{b \times v}(0,1)$ is regular E–optimal if and only if

$$\mathbf{X}'\mathbf{X} = \frac{b}{4(v-1)}\left(v\mathbf{I}_v + (v-2)\mathbf{1}_v\mathbf{1}_v'\right). \tag{2.1}$$

In Jacroux and Notz (1983), the method of construction of the regular E–optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{b \times v}(0,1)$ based on the incidence matrices of the balanced incomplete block designs is given. In this paper, for even v, we give new construction method of the regular E–optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{b \times v}(0,1)$ that widest the class of the designs given by Jacroux and Notz (1983). This method is based on the incidence matrices of two group divisible designs with the same association scheme.

3. Construction of the design matrix

Now, we recall the definition of the partially balanced incomplete block design with two associate classes given, for instance, in Clatworthy (1973).

An incomplete block design is said to be partially balanced with two associate classes if it satisfies the following requirements

(i) The experimental material is devided into b blocks of k units each, different treatments being applied to the units in the same block.

(ii) There are $v \ (> k)$ treatments each of which occurs in r blocks.

(iii) There can be established a relation of association between any two treatments satisfying the following requirements:

a) Two treatments are either first associates or second associates.

b) Each treatment has exactly q_α αth associates, $\alpha = 1,2$.

c) Given any two treatments which are αth associates, the number of treatments common to the βth associate of the first and the γth associate of the second is $p^{\alpha}_{\beta\gamma}$ and is independent of the pair of treatments we start with. Also $p^{\alpha}_{\beta\gamma} = p^{\alpha}_{\gamma\beta}$, $\alpha, \beta, \gamma = 1, 2$.

d) Two treatments which are αth associates occur together in exactly λ_α blocks, $\alpha = 1, 2$.

For a proper partially balanced incomplete block design $\lambda_1 \neq \lambda_2$. The numbers v, b, r, k, λ_1, λ_2 are called the parameters of the first kind, whereas the numbers q_α, $p^{\alpha}_{\beta\gamma}$, $\alpha, \beta, \gamma = 1, 2$ are called the parameters of the second kind.

A group divisible design is a partially balanced incomplete block design with two associate classes for which the treatments may be divided into m groups of s distinct treatments each, such that treatments that belong to the same group are first associates and two treatments belonging to different groups are second associates. For group divisible design it is clear that $v = ms$, $q_1 = s - 1$, $q_2 = s(m-1)$, $(s-1)\lambda_1 + s(m-1)\lambda_2 = r(k-1)$.

Based on two incidence matrices of group divisible designs with the same association scheme, we construct the regular E–optimal spring balance weighing design.

Let us consider the design matrix $\mathbf{X} = \begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\end{bmatrix}'$, where $\mathbf{N}_1$ and $\mathbf{N}_2$ are the incidence matrices of the group divisible designs with the same association scheme with parameters v, b_t, r_t, k_t, λ_{1t}, λ_{2t}, $t = 1,2$ and let

$$\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = \lambda. \tag{3.1}$$

Theorem 3.1 Let v be even. $\mathbf{X} \in \mathbf{\Phi}_{(b_1+b_2)\times v}(0,1)$ in the form $\mathbf{X} = [\mathbf{N}_1 \quad \mathbf{N}_2]'$, where $\mathbf{N}_1$ and $\mathbf{N}_2$ are the incidence matrices of the group divisible design with the same association scheme with the parameters $v,\ b_t,\ r_t,\ k_t,\ \lambda_{1t},\ \lambda_{2t},\ t = 1,2$, is the regular E–optimal spring balance weighing design if and only if the conditions

(a) $b_1 + b_2 = 2(r_1 + r_2)$

(b) $4\lambda(v-1) = (v-2)(b_1 + b_2)$

are fulfilled simultaneously.

Proof. For $\mathbf{X} \in \mathbf{\Phi}_{(b_1+b_2)\times v}(0,1)$ from the condition (2.1) we have

$$\mathbf{X}'\mathbf{X} = \mathbf{N}_1\mathbf{N}_1' + \mathbf{N}_2\mathbf{N}_2' = \frac{(b_1+b_2)v}{4(v-1)}\mathbf{I}_v + \frac{(b_1+b_2)(v-2)}{4(v-1)}\mathbf{1}_v\mathbf{1}_v'. \tag{3.2}$$

On the other hand, $\mathbf{N}_1\mathbf{N}_1' + \mathbf{N}_2\mathbf{N}_2' = (r_1 + r_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$. Thus (3.2) is satisfied if and only if $\lambda = \frac{(b_1+b_2)(v-2)}{4(v-1)}$ and we are given in the condition (b). Considering Theorem 2.1 and the equality $\frac{(b_1+b_2)v}{4(v-1)} = r_1 + r_2 - \lambda$ we obtain the condition (a). Hence the result.

Based on the book of Clatworthy (1973) we formulate theorems giving the parameters of group divisible designs having appropriate design numbers (for example R1).

Theorem 3.2 Let $v = 4$ and let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of the group divisible designs with the same association scheme with the parameters

De–sign	b_1	r_1	k_1	λ_{11}	λ_{21}	Sym–bol	b_2	r_2	k_2	λ_{12}	λ_{22}	Sym–bol
1	8	4	2	2	1	R1	10	5	2	1	2	R3
2	8	4	2	2	1	R1	16	8	2	2	3	R10
3	8	4	2	2	1	R1	4	2	2	0	1	SR1
4	10	5	2	3	1	R2	14	7	2	1	3	R7
5	10	5	2	3	1	R2	20	10	2	2	4	R17
6	10	5	2	3	1	R2	8	4	2	0	2	SR2
7	10	5	2	1	2	R3	14	7	2	3	2	R6
8	10	5	2	1	2	R3	20	10	2	4	3	R16
9	12	6	2	4	1	R4	12	6	2	0	3	SR3
10	14	7	2	5	1	R5	16	8	2	0	4	SR4
11	14	7	2	3	2	R6	16	8	2	2	3	R10
12	14	7	2	3	2	R6	4	2	2	0	1	SR1
13	14	7	2	1	3	R7	16	8	2	4	2	R9
14	16	8	2	6	1	R8	20	10	2	0	5	SR5
15	16	8	2	4	2	R9	20	10	2	2	4	R17
16	16	8	2	4	2	R9	8	4	2	0	2	SR2
17	16	8	2	2	3	R10	20	10	2	4	3	R16
18	18	9	2	5	2	R12	18	9	2	1	4	R13
19	18	9	2	5	2	R12	12	6	2	0	3	SR3
20	20	10	2	6	2	R15	16	8	2	0	4	SR4
21	20	10	2	4	3	R16	4	2	2	0	1	SR1

then $\mathbf{X} = \left[\mathbf{N}_1 \quad \mathbf{N}_2\right]' \in \mathbf{\Phi}_{(b_1+b_2)\times 4}(0,1)$ is the regular E–optimal spring balance weighing design.

Proof. This can be proved by checking that the parameters given above satisfy conditions (a) and (b) of Theorem 3.1.

Theorem 3.3 Let $v = 6$ and let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of the group divisible designs with the same association scheme with the parameters

De–sign	b_1	r_1	k_1	λ_{11}	λ_{21}	Sym–bol	b_2	r_2	k_2	λ_{12}	λ_{22}	Sym–bol
1	12	6	3	3	2	R43	18	9	3	3	4	R52
2	4	2	3	0	1	SR18	6	3	3	2	1	R42
3	4	2	3	0	1	SR18	16	8	3	4	3	R48
4	8	4	3	0	2	SR19	12	6	3	4	2	R44
5	12	6	3	0	3	SR20	18	9	3	6	3	R50
6	6	3	3	2	1	R42	14	7	3	2	3	R46
7	12	6	3	4	2	R44	18	9	3	2	4	R51
8	14	7	3	2	3	R46	16	8	3	4	3	R48

then $\mathbf{X} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \end{bmatrix}' \in \mathbf{\Phi}_{(b_1+b_2)\times 6}(0,1)$ is the regular E–optimal spring balance weighing design.

Proof. It is easily seen that the parameters given above satisfy conditions (a) and (b) of Theorem 3.1.

Theorem 3.4 Let $v = 8$ and let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of the group divisible designs with the same association scheme with the parameters

Design	b_1	r_1	k_1	λ_{11}	λ_{21}	Symbol	b_2	r_2	k_2	λ_{12}	λ_{22}	Symbol
1	12	6	4	2	3	SR38	16	8	4	4	3	R98
2	6	3	4	3	1	S6	8	4	4	0	2	SR36
3	12	6	4	6	2	S7	16	8	4	0	4	SR39
4	8	4	4	0	2	SR36	20	10	4	6	4	R103
5	12	6	4	0	3	SR37	16	8	4	6	3	R99
6	10	5	4	3	2	R97	18	9	4	3	4	R101

then $\mathbf{X} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \end{bmatrix}' \in \mathbf{\Phi}_{(b_1+b_2)\times 8}(0,1)$ is the regular E–optimal spring balance weighing design.

Proof. An easy computation shows that the parameters given above satisfy conditions (a) and (b) of Theorem 3.1.

Theorem 3.5 Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of the group divisible designs with the same association scheme with the parameters

No	v	b_1	r_1	k_1	λ_{11}	λ_{21}	Symbol	b_2	r_2	k_2	λ_{12}	λ_{22}	Symbol
1	10	8	4	5	0	2	SR52	10	5	5	4	2	R139
2	10	16	8	5	0	4	SR54	20	10	5	8	4	R142
3	12	10	5	6	5	2	S28	12	6	6	0	3	SR67
4	14	12	6	7	0	3	SR81	14	7	7	6	3	R177
5	16	14	7	8	7	3	S63	16	8	8	0	4	SR92
6	18	16	8	9	0	4	SR100	18	9	9	8	4	R197
7	20	18	9	10	9	4	S109	20	10	10	0	5	SR108

then $\mathbf{X} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \end{bmatrix}' \in \mathbf{\Phi}_{(b_1+b_2)\times v}(0,1)$ is the regular E–optimal spring balance weighing design.

Proof. The main idea of proof is showing that the parameters given above satisfy conditions (a) and (b) of Theorem 3.1.

4. Example

For $n = 14$ and $v = 8$, let us consider the class $\mathbf{\Phi}_{14\times 8}(0,1)$ of the design matrices of the spring balance weighing designs. Based on Theorem 3.4 there exist the group divisible block designs with the same association scheme and with the parameters $v = 8$, $b_1 = 6$, $r_1 = 3$, $k_1 = 4$, $\lambda_{11} = 3$, $\lambda_{21} = 1$ (S6) and $v = 8$, $b_2 = 8$, $r_2 = 4$, $k_2 = 4$, $\lambda_{12} = 0$, $\lambda_{22} = 2$ (SR36) given by the incidence matrices $\mathbf{N}_1$ and $\mathbf{N}_2$, where

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}, \mathbf{N}_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\text{association scheme } \begin{matrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{matrix}.$$

Hence $$\mathbf{X}' = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

The design $\mathbf{X} \in \mathbf{\Phi}_{14\times 8}(0, 1)$ is the regular E–optimal spring balance design and its maximal eigenvalue is equal 28.

References

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