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STOCHASTIC MODELING IN SELECTED BIOLOGICAL SYSTEMS

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Summary

The stochastic differential equation with colored white noise is used for the description of the dynamics of some selected biological communities. The deterministic models of single species population models are transformed to the stochastic models. The parametric identification of these models is done by the maximum likelihood method. The effectiveness of the estimation procedures is proved by Monte Carlo simulations as well as checked on the real data for Whooping Crane population.

Key words and phrases: population dynamics, stochastic differential equation, colored white noise, system identification

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1. Introduction

Biological systems such as the communities of animals and plants react on the changes of existence conditions, that is to say on environment actions and own states. That is why the description of the systems dynamics, connected with the selection of a mathematical model and its verification, is still open problem. The solution of this problem is never unique. Brännaström and Sumpter (2006) stated that an effective population dynamics model has to take into account aftereffects and joint effects of different external and internal factors as far as the exact nature of the noise in a population is never known and can be explained only as demographic and environmental noises. It is necessary to notice that the deterministic models presented by the ordinary or even the partial differential equations do not allow to analyze all kinds of the environmental fluctuations, especially, if they are significant. To improve the model, as it was done by Allen and Allen (2003), it is possible to involve more variables or to change a class of the models. Let us consider several possible transformations.

The first mathematical description of the growth of a certain population was proposed by Malthus (1802). It is a linear differential equation given as (see Murray, 2006)

$$dY_t = \mu Y_t dt \,,\; Y(t_0) = Y_0 \,, \tag{1.1}$$

where $Y_t \in \mathrm{R}$ is the size of the population (the numbers of individuals), $\mu \subseteq \mathrm{R}$ is the coefficient of the natural growth of the population, $t \in [t_0, T]$, $t_0 < T < \infty$. The unbounded growth model (1.1) is valid only in a few cases, as for example food resource, and describes a potential possibility of medium for the certain population as far as in real life situations the population size is bounded by environment capacity.

Improvements of Malthus model introduced by Pearl and Reed (1920) gave possibilities to describe a sufficiently real growth law for many populations of microorganisms, animals and humans by non–linear differential equation (see Murray, 2006)

$$dY_t = \mu Y_t \left(1 - \frac{Y_t}{K}\right) dt \,,\; Y(t_0) = Y_0 \,, \tag{1.2}$$

where the multiplier $\left(1 - \frac{Y_t}{K}\right)$ represents the medium resistivity for the increase of the population, $K \neq 0$ ($K \subseteq \mathrm{R}$) is the environment capacity.

To describe the self–intoxication process of one–species population by the waste products of its own metabolism Volterra (1931) added integral delay to the Verhulst–Pearl model and got the following integral–differential equation (see Murray, 2006)

$$dY_t = \mu Y_t \left(1 - \frac{Y_t}{K} - \int_0^t r(t-\tau)\, d\tau\right) dt \,,\; Y(t_0) = Y_0 \,, \tag{1.3}$$

where $r(\cdot)$ is the hereditary function which shows the influence of the prehistory at the instant on the population dynamics, moreover, the integral term introduces a process aftereffect.

It is possible to notice that mentioned models are nested, namely we can come from the model (1.3) to the model (1.1) still dealing with the self–intoxication process. It depends only on the values of the parameters. These models are idealistic and do not reflect influences of the external environment to the systems. In order to improve these models and to take into account random disturbances, which are not necessary small, we can suppose that system dynamics corresponds to some stochastic process.

Let $(\Omega, \mathsf{F}, \mathsf{P})$ be a stochastic basis satisfying the usual conditions (see Jazwinski, 2007). Let $Y:\Omega \to \mathsf{R}$ be the size of the population, such that $\omega \in \Omega$, $Y(\omega) = y$, and $\{v(t), t \in [t_0, T]\}$ be a continuous stochastic process, defined on $(\Omega, \mathsf{F}, \mathsf{P})$, such that its mean value function $\mathsf{E}[v(t)] = 0$ for every $t \in [t_0, T]$ and $v(t_0) = 0$ ($\mathsf{E}[\cdot]$ denotes the expectation operator). In addition, we suppose that this process has the following properties:

(1) its increments $v(t+h) - v(\tau + h)$ are independent and stationary for every $t > \tau \in [t_0, T]$ and every $h > 0$;

(2) the mean square continuity of $v(t)$ for every $t \in [t_0, T]$;

(3) the regularity conditions, i.e. $\mathsf{E}\left[\left|v(t_0)\right|^2\right] < \infty$ and $\operatorname{var}\left[v(t+h) - v(\tau+h)\right] < \infty$ for every $t > \tau \in [t_0, T]$ and every $h > 0$ ($\operatorname{var}[\cdot]$ denotes the variance operator).

Many different processes meet these conditions (see Jazwinski, 2007). To stay with this idea it is reasonable to use some stochastic process as far as it gives possibilities to describe different situations of external influences as long–range or short–range dependences, aftereffects or absence of named situations. We use the colored white noise, which increments are $\xi_t (dt)^H$ ($\xi_t \sim \mathsf{N}(0,1)$ i.i.d., $0 < H < 1$ is an fractal index), as a model of the process $v(t)$ and rewrite the model (1.1) as the linear stochastic differential equation (SDE)

$$dY_t = \mu Y_t + \sigma Y_t \xi_t (dt)^H, \; Y(t_0) = Y_0, \tag{1.4}$$

and models (1.2) and (1.3) as the non–linear SDE

$$dY_t = \left(\mu Y_t - \beta Y_t^2\right) dt + \sigma Y_t \xi_t \left(dt\right)^H ,\ Y\left(t_0\right) = Y_0 , \qquad (1.5)$$

where μ, $\beta = \frac{\mu}{K}$ and σ are some unknown parameters.

Since (1.1) is called the unlimited growth model and (1.2), (1.3) are limited growth models, further on we call the model (1.4) as the stochastic unlimited growth model and model (1.5) as the stochastic limited growth model. The aim of this paper is to show the method for the parameters estimation of the SDEs (1.4) and (1.5).

2. Parameter identification method

2.1. Problem formulation

The continuous stochastic processes $Y = \left\{Y_t, t \in \left[t_0, T\right]\right\}$ are assumed to be the unique strong solution of the SDEs (1.4) and (1.5), which in general form can be written as follows

$$dY_t = f\left(Y_t, \boldsymbol{\theta}\right) dt + g\left(Y_t, \boldsymbol{\theta}\right) \xi_t \left(dt\right)^H ,\ Y\left(t_0\right) = Y_0 , \qquad (2.1)$$

where $f\left(Y_t, \boldsymbol{\theta}\right): \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}$ and $g\left(Y_t, \boldsymbol{\theta}\right): \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}$ are some function, $\boldsymbol{\theta}$ is a vector of n unknown parameters.

We suppose that for the process Y there are real valued observations $X_0, X_1, ..., X_N \in \mathrm{R}$ made at the equidistant discretization times $t_0 < t_1 < ... < t_N = T$. We also remark that the stochastic difference equation

$$X_{k+1} = X_k + f\left(X_k, \boldsymbol{\theta}\right)\Delta + g\left(X_k, \boldsymbol{\theta}\right)\xi_k \Delta^H , \qquad (2.2)$$

(where $\Delta = t_{k+1} - t_k$ for $k = 0,1,...,N$ and $X_0 = Y_0$) is an discrete analog of the SDE (2.1) with accuracy

$$\varepsilon(\Delta) = \mathrm{E}\left(\left|Y_T - X_T^{\Delta}\right|\right) \le C\Delta^{1/2} , \qquad (2.3)$$

where positive constant C does not depend on Δ. The observations contain the information about the parameters vector $\boldsymbol{\theta} \in \Theta \subseteq \mathsf{R}$ (where Θ specifies the set of allowable values for the parameters) that we wish to estimate. Since the colored white noise, used in the model (2.1), is Markov process (see Jazwinski, 2007), the maximum likelihood estimation of the parameters $\boldsymbol{\theta}$ is one of the possible solutions to the problem. This requires the construction of the likelihood function $\mathsf{L}(\boldsymbol{\theta})$ which is based on the probability density function (pdf) known for each transition (see Filatova and Grzywaczewski, 2007b or Hurn et al. 2003).

Jumarie (2007) showed that first and second moments of the increments of the process X_k, $k = 0,1,...,N$, are $\mathsf{E}\{\Delta X \mid X, k\} = f(X_k, \boldsymbol{\theta})\Delta$ and $\mathsf{E}\left\{(\Delta X)^2 \mid X, k\right\} = g^2(X_k, \boldsymbol{\theta})\Delta^{2H}$. The pdf of (X_{k+1}, t_{k+1}) for a process (2.2) starting at (X_k, t_k) is

$$pdf\left(X_{k+1}, t_{k+1} \mid X_k, t_k; \boldsymbol{\theta}\right) = \frac{1}{\sqrt{2\pi g^2(X_k, \boldsymbol{\theta})\Delta^H}} \exp\left\{-\frac{\left(X_{k+1} - X_k - f(X_k, \boldsymbol{\theta})\Delta\right)^2}{2g^2(X_k, \boldsymbol{\theta})\Delta^{2H}}\right\}. \quad (2.4)$$

The joint density corresponds to the likelihood function

$$\mathsf{L}(\boldsymbol{\theta}) = pdf\left(X_0, t_0; \boldsymbol{\theta}\right) \prod_{k=0}^{N-1} pdf\left(X_{k+1}, t_{k+1} \middle| X_k, t_k; \boldsymbol{\theta}\right). \quad (2.5)$$

Finally, the parameters estimation task for the SDE (2.1) consists in

$$\mathsf{L}\left(\hat{\boldsymbol{\theta}}\right) = \sup_{\boldsymbol{\theta} \in \Theta} \mathsf{L}(\boldsymbol{\theta}), \quad (2.6)$$

where $\hat{\boldsymbol{\theta}}$ is the estimate of $\boldsymbol{\theta}$, $\sup(\cdot)$ is the least upper bound of $\mathsf{L}(\boldsymbol{\theta})$ over all $\boldsymbol{\theta} \in \Theta$.

2.2. Stochastic unlimited growth model

For the equation (1.4) functions are $f(X_k, \boldsymbol{\theta}) = \mu X_k$ and $g(X_k, \boldsymbol{\theta}) = \sigma X_k$, where $\boldsymbol{\theta} = [\mu, \beta]^{\mathsf{T}}$. So, the maximum likelihood function is

$$\mathsf{L}(\mu,\sigma)=\prod_{k=0}^{N-1}\frac{1}{\sqrt{2\pi\sigma^2X_k^2\Delta^H}}\exp\left\{-\frac{\left(X_{k+1}-X_k-\mu X_k\Delta\right)^2}{2\sigma^2X_k^2\Delta^{2H}}\right\}.$$

For the simplification we replace product function by sums

$$\begin{aligned}-\log\mathsf{L}(\mu,\sigma)=\frac{1}{2}\sum_{k=0}^{N-1}\frac{\left(X_{k+1}-X_k-\mu X_k\Delta\right)^2}{\sigma^2X_k^2\Delta^{2H}}+\frac{1}{2}\sum_{k=0}^{N-1}\log X_k^2\\+\frac{N}{2}\left(\log\left(\sigma^2\right)+\log\left(2\pi\Delta^{2H}\right)\right).\end{aligned}\tag{2.7}$$

This simplification means that the task (2.6) has to be rewritten as follows

$$\mathsf{L}\left(\hat{\boldsymbol{\theta}}\right)=\inf_{\boldsymbol{\theta}\in\Theta}\left(-\log\mathsf{L}(\boldsymbol{\theta})\right),\tag{2.8}$$

where $\inf(\cdot)$ is the greatest lower bound of $\mathsf{L}(\boldsymbol{\theta})$ over all $\boldsymbol{\theta}\in\Theta$.

The optimal solution of (2.8) can be determined if partial derivatives of (2.7) are zero, namely

$$-\frac{\partial\log\mathsf{L}(\mu,\sigma)}{\partial\mu}=\frac{\Delta}{\sigma^2\Delta^{2H}}\sum_{k=0}^{N-1}\frac{\mu X_k\Delta+X_k-X_{k+1}}{X_k}\equiv 0,$$

$$-\frac{\partial\log\mathsf{L}(\mu,\sigma)}{\partial\sigma}=-\frac{1}{4\sigma^3}\sum_{k=0}^{N-1}\frac{\left[X_{k+1}-X_k-\mu X_k\Delta\right]^2}{X_k^2\Delta^{2H}}+\frac{N}{4}\equiv 0.$$

Finally, we got following estimates of the parameters $\hat{\boldsymbol{\theta}}=\left[\hat{\mu},\hat{\sigma}\right]^{\mathsf{T}}$

$$\hat{\mu}=\frac{1}{N\Delta}\sum_{k=0}^{N-1}\frac{X_{k+1}-X_k}{X_k},\tag{2.9}$$

$$\hat{\sigma}^2=\frac{1}{\Delta^{2H}N}\sum_{k=0}^{N-1}\frac{\left[X_{k+1}-X_k-\hat{\mu}X_k\Delta\right]^2}{X_k^2}.\tag{2.10}$$

2.3. Stochastic limited growth model

For the model (1.5) parameters functions are $f\left(X_k,\boldsymbol{\theta}\right)=\mu X_k-\beta X_k^2$ and $g\left(X_k,\boldsymbol{\theta}\right)=\sigma X_k$, where $\boldsymbol{\theta}=\left[\mu,\beta,\sigma\right]^{\mathsf{T}}$. Using the same arguments as in the previous case, we can write the negative log–likelihood function as

$$-\log \mathsf{L}\left(\mu,\beta,\sigma\right)=\frac{1}{2}\sum_{k=0}^{N-1}\frac{\left(X_{k+1}-X_k-\left(\mu X_k-\beta X_k^2\right)\Delta\right)^2}{\sigma^2 X_k^2\Delta^{2H}}+\frac{1}{2}\sum_{k=0}^{N-1}\log X_k^2 \\ +\frac{N}{2}\left(\log\left(\sigma^2\right)+\log\left(2\pi\Delta^{2H}\right)\right), \tag{2.11}$$

which allows to find partial derivatives

$$-\frac{\partial\log \mathsf{L}\left(\mu,\beta,\sigma\right)}{\partial\mu}=\frac{\Delta}{\sigma^2\Delta^{2H}}\sum_{k=0}^{N-1}\left(\frac{X_k-X_{k+1}}{X_k}+\left(\mu-\beta X_k\right)\Delta\right), \tag{2.12}$$

$$-\frac{\partial\log \mathsf{L}\left(\mu,\beta,\sigma\right)}{\partial\beta}=\frac{\Delta}{\sigma^2\Delta^{2H}}\sum_{k=0}^{N-1}\left(X_{k+1}-X_k-\left(\mu-\beta X_k\right)\Delta\right), \tag{2.13}$$

$$-\frac{\partial\log \mathsf{L}\left(\mu,\beta,\sigma\right)}{\partial\sigma}=-\frac{1}{4\sigma^3}\sum_{k=0}^{N-1}\frac{\left[X_{k+1}-X_k-\left(\mu-\beta X_k\right)X_k\Delta\right]^2}{X_k^2\Delta^{2H}}+\frac{N}{\sigma}. \tag{2.14}$$

Setting partial derivatives (2.12) – (2.14) equal to zero allows to find the optimal solution of (2.7), namely the following estimates $\hat{\boldsymbol{\theta}}=\left[\hat{\mu},\hat{\beta},\hat{\sigma}\right]^{\mathsf{T}}$

$$\hat{\mu}=\frac{1}{N}\left\{\hat{\beta}\sum_{k=0}^{N-1}X_k+\frac{1}{\Delta}\sum_{k=0}^{N-1}\frac{X_{k+1}-X_k}{X_k}\right\}, \tag{2.15}$$

$$\hat{\beta}=\frac{\hat{\mu}\sum_{k=0}^{N-1}X_k-\frac{1}{\Delta}\sum_{k=0}^{N-1}\left(X_{k+1}-X_k\right)}{\sum_{k=0}^{N-1}X_k^2} \tag{2.16}$$

and

$$\hat{\sigma}^2 = \frac{1}{\Delta^{2H} N} \sum_{k=0}^{N-1} \frac{\left[X_{k+1} - X_k - \left(\hat{\mu} X_k - \hat{\beta} X_k^2 \right) \Delta \right]^2}{X_k^2} . \qquad (2.17)$$

Remark 2.1. Since in both cases the estimates $\hat{\boldsymbol{\theta}}$ contain $\hat{\sigma}^2$ instead of $\hat{\sigma}$ and are found on the basis of (2.2), to determine the sing of the estimate of the parameter σ the optimization of (2.7) has to be done under constrains (2.3). Moreover, the systems (2.9) – (2.10) and (2.15) – (2.17) require the iterative solution, which can be done by an appropriate numerical algorithm such as the $L\pi_\tau$ method (see Sobol, 1979).

Remark 2.2. For the models (1.4) and (1.5) the maximum likelihood estimation theory allows us to derive asymptotic properties of $\hat{\boldsymbol{\theta}}$, i.e. the estimates are a.s. consistent and asymptotically normal (for the details see Xiao et al. 2011).

3. Simulation experiments

To illustrate the efficiency of the estimates (2.9) – (2.10) and (2.15) – (2.17), we completed some numerical experiments using Monte Carlo simulations. The simulation procedure was as follows:

1) set step interval $\Delta = 10^{-4}$, $t \in [0,5]$, $C = 1.0$, starting values of the parameters $\boldsymbol{\theta}^s = [0.0, 0.0]^\mathsf{T}$ in (1.4) and $\boldsymbol{\theta}^s = [0.0, 0.0, 0.0]^\mathsf{T}$ in (1.5), the initial value $Y_0 = X_0 = 1$;
2) generate sample path of the standard Brownian motion for selected Δ;
3) use this sample path for each value $H \in \{0.1, 0.5, 0.8\}$ to generate normalized colored white noise by means of the fast Fourier transform (see Paxson, 1997);
4) setting the true value of the parameters, namely $\boldsymbol{\theta} = [2.0, 0.5]^\mathsf{T}$ in the model (1.4) and $\boldsymbol{\theta} = [2.0, 0.01, 0.25]^\mathsf{T}$ in the model (1.5) use (2.2) to obtain synthetic data Y for each sample path of colored white noise;

5) for models (1.4) and (1.5) find the estimates $\hat{\boldsymbol{\theta}}$, generating sample paths of colored white noise as in steps 2) – 3) and getting X by (2.2) for $\Delta \in \{10^{-1}, 10^{-2}, 10^{-3}\}$ and repeating this step $N^{SP} \in \{10, 100\}$ times.

The averaged estimates as well as their standard deviations are listed in Table 1 and Table 2.

Table 1. Monte Carlo experiments for the model (1.4)

Δ	H	N^{SP}	$\bar{\hat{\mu}} \pm std_{\hat{\mu}}$	$\bar{\hat{\sigma}} \pm std_{\hat{\sigma}}$
10^{-1}	0.1	10	1.7668 ± 0.1161	0.3366 ± 0.0382
	0.5		1.7400 ± 0.1685	0.4080 ± 0.0200
	0.8		1.7771 ± 0.0651	0.3263 ± 0.0417
10^{-2}	0.1		1.8173 ± 0.0822	0.4524 ± 0.0247
	0.5		1.9097 ± 0.3223	0.4830 ± 0.0094
	0.8		1.8403 ± 0.1630	0.4328 ± 0.0073
10^{-3}	0.1		1.9741 ± 0.0905	0.4586 ± 0.0648
	0.5		2.0418 ± 0.2006	0.4377 ± 0.0044
	0.8		1.9678 ± 0.0947	0.4666 ± 0.0015
10^{-1}	0.1	100	1.8476 ± 0.0931	0.4741 ± 0.0270
	0.5		1.8470 ± 0.1267	0.4109 ± 0.0198
	0.8		1.8361 ± 0.0871	0.3930 ± 0.0210
10^{-2}	0.1		2.0328 ± 0.0759	0.4628 ± 0.0198
	0.5		2.0199 ± 0.1209	0.4823 ± 0.0014
	0.8		2.0099 ± 0.0723	0.4529 ± 0.0047
10^{-3}	0.1		1.9689 ± 0.0873	0.4688 ± 0.0588
	0.5		1.9692 ± 0.1223	0.4788 ± 0.0039
	0.8		2.0022 ± 0.0740	0.4752 ± 0.0010

As we can see from the results in all cases estimates received on the basis of approximation with the stepsize $\Delta = 10^{-1}$ demonstrated significant left–hand side bias despite small values of standard deviations. Decrease of the stepsize and increase of sample paths number for all values of H improved the estimates and gave unbiased results with standard deviations less then 5% for all cases. Therefore, it is possible to conclude that the most important impact during parameters estimation of the processes (1.4) and (1.5) is the stepsize selection. Smaller values of the stepsize guaranties unbiased estimators of the SDEs parameters with relatively small standard deviation. In the most cases for $H \in \{0.5, 0.8\}$ the standard deviation for the parameter σ in both models was smaller than for $H = 0.1$. This fact can be explained by the properties of colored white noise (see Xiao et al. 2011).

Table 2. Monte Carlo experiments for the model (1.5)

Δ	H	N^{SP}	$\bar{\hat{\mu}} \pm std_{\hat{\mu}}$	$\bar{\hat{\beta}} \pm std_{\hat{\beta}}$	$\bar{\hat{\sigma}} \pm std_{\hat{\sigma}}$
10^{-1}	0.1	10	1.8844 ± 0.0539	0.0084 ± 0.0002	0.2035 ± 0.0052
	0.5		1.7833 ± 0.0865	0.0080 ± 0.0001	0.2682 ± 0.0069
	0.8		1.7985 ± 0.0563	0.0079 ± 0.0002	0.2292 ± 0.0020
10^{-2}	0.1		2.0755 ± 0.0261	0.0091 ± 0.0001	0.1806 ± 0.0055
	0.5		1.9404 ± 0.0435	0.0100 ± 0.0001	0.2404 ± 0.0052
	0.8		1.9850 ± 0.0234	0.0099 ± 0.0002	0.2368 ± 0.0032
10^{-3}	0.1		1.9550 ± 0.0307	0.0088 ± 0.0000	0.2479 ± 0.0166
	0.5		1.9885 ± 0.0184	0.0095 ± 0.0001	0.2221 ± 0.0089
	0.8		2.0040 ± 0.0243	0.0090 ± 0.0001	0.0243 ± 0.0037
10^{-1}	0.1	100	1.8032 ± 0.0581	0.0071 ± 0.0000	0.1869 ± 0.0067
	0.5		1.8320 ± 0.0506	0.0073 ± 0.0000	0.2175 ± 0.0036
	0.8		1.7945 ± 0.0614	0.0071 ± 0.0001	0.2232 ± 0.0025
10^{-2}	0.1		1.9723 ± 0.0276	0.0085 ± 0.0001	0.1851 ± 0.0045
	0.5		1.9900 ± 0.0277	0.0098 ± 0.0001	0.2446 ± 0.0033
	0.8		1.9950 ± 0.0219	0.0092 ± 0.0000	0.2280 ± 0.0011
10^{-3}	0.1		2.0028 ± 0.0222	0.0096 ± 0.0000	0.2293 ± 0.0147
	0.5		1.9756 ± 0.0342	0.0098 ± 0.0000	0.2198 ± 0.0093
	0.8		1.9915 ± 0.0183	0.0101 ± 0.0000	0.2333 ± 0.0037

4. The Whooping Crane population model

The Whooping Crane (Grus Americana) is one of endangered species since 1938. During recovery period (1938 – 2005) the population which was observed in Aransas National Wild–life Refuge and Wood Buffalo National Park grew from 18 to 217 individuals. Annual data (coded as t) corresponding to number of individuals (coded as $X(t)$) observed in October are shown in Fig. 1.

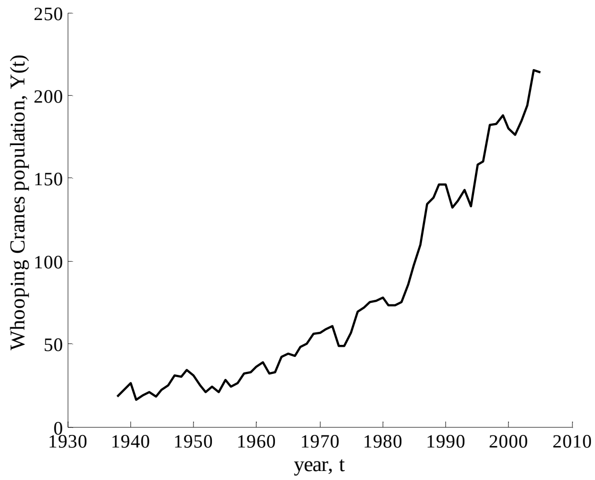

Fig. 1. Annual observations of Whooping Cranes population counted every October

The problem of this population model selection was widely studied starting with Dennis et al. (1991). In the book of Allen (2007) the nonparametric method was used to find the parameters of the model, proposed for the description of the dynamics of Whooping Cranes population. This model was given by following SDE (Allen 2007, p. 122)

$$dX_t = 0.0361 X_t\, dt + \sqrt{0.579 X_t}\, dB_t\,,\ X_0 = 18\,, \tag{4.1}$$

where dB_t is an increment of the ordinary Brownian motion.

We used the model (4.1) for comparison with our maximum likelihood approach. For this purpose, parameter H was estimated by semiparametric method (Filatova and Grzywaczewski, 2007a). Since $H = 0.5602$ and $p(H \neq 0.5) < 0.002$ that indicates the long–range dependence process in the population dynamics. Using recommendation of previous subsection and recoding time variable as $t_{1938} = 0$, $t_{1939} = 0.0125$, $t_{1940} = 0.0250$,..., we estimated parameters of the models (1.4) and (1.5) and got following results:

$$dX_t = 3.0890\, X_t dt + 1.8743\, X_t \xi_t (dt)^{0.5602}\,, \tag{4.2}$$

$$dX_t = \left(3.7641\, X_t - 0.01121\, X_t^2\right) dt + 0.1270\, X_t \xi_t (dt)^{0.5602}\,. \tag{4.3}$$

To compare the models (4.1) – (4.3) we used Q statistics as a goodness–of–fit test for an SDE model (see Allen (2007), pp. 183 – 185). This test statistics is approximately distributed as a χ^2 random variable with $M-n$ degrees of freedom, where M stands for the number of the simulations used to find Q and n is a number of the parameters in the model. If $p\left(\chi^2\left(M\right)\geq Q\right)$ is smaller than a preset level of significance α that means a lack–of–fit of the SDE model with data. Setting $\alpha = 0.05$ and $M = 8$, we got $p\left(\chi^2\left(6\right)\geq 4.1019\right)< 0.651$ for the model (4.1), $p\left(\chi^2\left(6\right)\geq 9.0502\right)< 0.223$ for the model (4.2), and $p\left(\chi^2\left(5\right)\geq 4.0916\right)< 0.723$ for the model (4.3). As we can see all the models can be used for the population description. However, taking into account long–range dependences with the appropriate model selection allows to get better fit to the data.

5. Conclusion

We studied the possibilities of stochastic modeling in single species population. We proposed the stochastic differential equation with colored white noise, which as the model gives more flexibility to describe complexity of biological systems. To find parameters of the stochastic limited and unlimited growth models we created maximum likelihood estimators. Numerical simulations and comparison studies showed the effectiveness of our methodology.

The short–range and long–range dependent processes influence biological systems and have to be taken into account in practical applications as it was shown in the optimal control task for the fishery (see Filatova et al. 2010). In the future we will try to combine the ideas presented here with ideas of Xiao et al. (2011) to find the estimation method for the SDE with mixed fractional Brownian motion.

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