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ROBUSTNESS OF OPTIMAL CHEMICAL BALANCE WEIGHING DESIGNS WITH CORRELATED ERRORS

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Summary

The estimation problem of individual weights of objects using the chemical balance weighing design under the restriction on the number of times in which each object is weighted is considered. The additional assumption is that the errors have the same variances and they are equal correlated. The conditions under which the existence of an optimal chemical balance weighing design for $p=v$ objects implies the existence of an optimal chemical balance weighing design for $p=v+1$ objects are given. Under these assumptions a new construction methods for the design matrix of the optimal chemical balance weighing design based on the incidence matrices of balanced block designs are given.

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1. Introduction

Let us suppose that we want to estimate the weights of p objects by weighing them n times on a chemical balance, $p \leq n$. If j th object weights

w_j, then the expected weight y_i of the objects on the left pan minus one on the right pan is

$$\mathrm{E}(y_i) = \sum_{j=1}^{p} x_{ij} w_j, \; i = 1,2,..., n,$$

where $x_{ij} = 1$ if the object j th is on the left pan in the weighing i, $x_{ij} =$ -1 if it is on the right pan and $x_{ij} = 0$ if it is not included in the weighing i. The matrix $\mathbf{X} = (x_{ij})$ is the weighing design matrix. $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$, where $\mathbf{\Phi}_{n\times p}(-1,0,1)$ denotes the class of $n \times p$ matrices having entries $x_{ij} =$ -1, 0 or 1. Let $\mathbf{e}$ be the vector of random errors. We assume that there are not systematic errors, they have the same variances and they are equal correlated according to the matrix

$$\mathbf{G} = \mathrm{g}\left[(1 - \rho)\mathbf{I}_n + \rho \mathbf{1}_n \mathbf{1}_n^{'}\right], \; g > 0, \; \rho \in \left(\frac{-1}{n-1}, 0\right], \tag{1.1}$$

i.e. $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{E}(\mathbf{ee}^{'}) = \sigma^2 \mathbf{G}$.

The matrix $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is the information matrix of the design. If $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, then weighted least square estimator of $\mathbf{w} = (w_1, w_2,..., w_p)^{'}$ is given by $\hat{\mathbf{w}} = \left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$ and the variance matrix of $\hat{\mathbf{w}}$ is of the form $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$. Under the assumption that all weights w_j are estimable, the problem we face with in this situation, is to construct the design matrix in such a way, that the best linear unbiased estimators of the weights are optimal according to the given optimality criterion.

Hotelling (1944) studied the estimation problem for the case $\mathbf{G} = \mathbf{I}_n$ and under the assumption that all objects are included in each measurement operation. The problems related to the determining unknown measurements of objects in the model of the optimal chemical balance weighing design is also considered in Raghavarao (1971), Banerjee (1975), Shah and Sinha (1989). The generalization of above problem was presented in Ceranka and Katulska (1998). Authors gave the necessary and sufficient conditions under which the lower bound of the variances of estimators was attained for the model of optimal chemical balance weighing design with equal correlated errors under an assumption that the elements of the matrix $\mathbf{X}$ are equal to -1 or 1, only. The similar problem was considered in Ceranka and Graczyk (2003, 2004) under the assumption that the elements of the design matrix $\mathbf{X}$ are equal to -1, 0 or 1. They gave the lower bound of the variance of each of the estimators and the definition of the optimal design.

Theorem 1.1. In the nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ under the assumption $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$ and with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), the variance of the estimated measurements of objects $\hat{\mathbf{w}}$ can be no less then

$$\mathrm{Var}(\hat{w}_j) \geq \sigma^2 \frac{g(1-\rho)}{m}, \quad j = 1,2,..., p, \tag{1.2}$$

where m is the number of elements equal to -1 or 1 in the j th column of $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$, $\left(m = \sum_{i=1}^{n} x_{ij}^2\right)$.

Definition 1.1. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ under the assumption $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$ and with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is optimal if the variance of each $\hat{w}_j$, $j = 1,2,..., p$, attains the lower bound given in Theorem 1.1.

The variance of estimators in optimal chemical balance weighing design is equal to

$$\mathrm{Var}(\hat{w}_j) = \sigma^2 \frac{g(1-\rho)}{m}, \quad j = 1,2,..., p. \tag{1.2}$$

Ceranka and Graczyk (2004) gave the necessary and sufficient conditions under which the chemical balance weighing design is optimal in the following Theorem.

Theorem 1.2. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ under the assumption $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$ and with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is optimal if and only if

$$\mathbf{X}'\mathbf{X} = m\mathbf{I}_p. \tag{1.3}$$

In this paper we study method of construction of an optimal chemical balance weighing design in the case $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ with the assumption $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$. This method uses the incidence matrices of balanced incomplete block design, balanced bipartite block design and ternary balanced block design for v treatments to form the design matrix of optimal chemical balance weighing design for $p = v+1$ objects.

2. Optimal chemical balance weighing design for $p = v+1$ objects

Now, let assume that $\mathbf{X}_h \in \mathbf{\Phi}_{n_h\times v}(-1,0,1)$ be the matrix of chemical balance weighing design, $h = 1,2$. We define the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$, where $n = 2(n_1 + n_2)$, of the chemical balance weighing design for $p = v+1$ objects as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{1}_{n_1} \\ -\mathbf{X}_1 & -\mathbf{1}_{n_1} \\ \mathbf{X}_2 & \mathbf{0}_{n_2} \\ -\mathbf{X}_2 & \mathbf{0}_{n_2} \end{bmatrix} \qquad (2.1)$$

where $\mathbf{1}_{n_1}$ is the $n_1 \times 1$ vector of units and $\mathbf{0}_{n_2}$ is the $n_2 \times 1$ vector of zeros.

Theorem 2.1. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is optimal if and only if

$$\mathbf{X}_1'\mathbf{1}_{n_1} = \mathbf{0}_v \quad \text{and} \qquad (2.2)$$

$$\mathbf{X}_1'\mathbf{X}_1 + \mathbf{X}_2'\mathbf{X}_2 = n_1\mathbf{I}_v. \qquad (2.3)$$

Proof. For $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by (2.1)

$$\mathbf{X}'\mathbf{X} = \begin{bmatrix} 2(\mathbf{X}_1'\mathbf{X}_1 + \mathbf{X}_2'\mathbf{X}_2) & 2\mathbf{X}_1'\mathbf{1}_{n_1} \\ 2\mathbf{1}_{n_1}'\mathbf{X}_1 & 2n_1 \end{bmatrix}. \qquad (2.4)$$

According to Theorem 1.2 the design is optimal if and only if $\mathbf{X}^{'}\mathbf{X} = m\mathbf{I}_{v+1}$. Then from (2.4) we have (2.2) and (2.3), which complete the proof.

In the consideration given above, we determine the optimal chemical balance weighing design for the estimation of unknown measurements of objects. The optimal design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ is the same for any ρ, $\rho \in \left(\frac{-1}{n-1}, 0\right]$, i.e. this design is robust for different ρ. The results given in the above Theorem imply the next corollary.

Corollary 2.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ under the assumption $\mathbf{X}^{'}\mathbf{1}_n = \mathbf{0}_p$ and with the covariance matrix of errors $\sigma^2\mathbf{I}_n$, is optimal for the estimated unknown measurements of objects if and only if such design is optimal for the estimated unknown measurements with the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1).

3. Balanced block designs

In this section we remind the definitions of balanced incomplete block design given by Raghavarao and Padgett (2005), balanced bipartite weighing design given in Huang (1976) and ternary balanced block design given in Billington (1984).

A balanced incomplete block design there is an arrangement of v treatments in b blocks, each of size k in such a way, that each treatment occurs at most once in each block, occurs in exactly r blocks and each pair of treatments occurs together in exactly λ blocks. The integers v, b, r, k, λ are called the parameters of the balanced incomplete block design. Let $\mathbf{N}$ be the incidence matrix of such a design. The parameters satisfy the following identities $vr = bk$, $\lambda(v-1) = r(k-1)$, $\mathbf{N}\mathbf{N}^{'} = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v^{'}$.

A balanced bipartite block design there is an arrangement of v treatments into b blocks, such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, where $k = k_1 + k_2$. Each treatment appears in r blocks. Each pair of treatments from different subblocks appears together in λ_1 blocks and each pair of treatments from the same subblock appears together in λ_2 blocks. The integers v, b, r,

k_1, k_2, λ_1, λ_2 are called the parameters of the balanced bipartite block design. Let $\mathbf{N}^*$ be the incidence matrix of such a design. The parameters satisfy the following equalities

$$vr = bk\,,\; b = \frac{\lambda_1 v(v-1)}{2k_1k_2},\; \lambda_2 = \frac{\lambda_1(k_1(k_1-1)+k_2(k_2-1))}{2k_1k_2},$$

$$r = \frac{\lambda_1(v-1)(k_1+k_2)}{2k_1k_2},\quad \mathbf{N}^*\mathbf{N}^{*'} = (r-\lambda_1-\lambda_2)\mathbf{I}_v + (\lambda_1+\lambda_2)\mathbf{1}_v\mathbf{1}_v^{'}.$$

A ternary balanced block design is defined as the design consisting of b blocks, each of the size k, chosen from a set of objects of size v, in such a way that each of the treatments occurs r times altogether and 0, 1 or 2 times in each block (2 appears at least ones) and each of the distinct pairs appears λ times. Any ternary balanced block design is regular, that means, each treatment occurs once in ρ_1 blocks and twice in ρ_2 blocks, where ρ_1 and ρ_2 are constant for the design. Let $\mathbf{N}$ be the incidence matrix of the ternary balanced block design. It is straightforward to verify that

$$vr = bk\,,\qquad r = \rho_1 + 2\rho_2\,,\qquad \lambda(v-1) = \rho_1(k-1) + 2\rho_2(k-2) = r(k-1) - 2\rho_2\,,$$

$$\mathbf{N}\mathbf{N}^{'} = (\rho_1 + 4\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v^{'} = (r + 2\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v^{'}.$$

4. Construction of the design matrix

Let $\mathbf{N}_i$ be the incidence matrix of ternary balanced block design with the parameters v, b_h, r_h, k_h, λ_h, ρ_{1h}, ρ_{2h}, $h = 1,2$. Now, we define matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1), where $\mathbf{X}_h = \mathbf{N}_h^{'} - \mathbf{1}_{b_i}\mathbf{1}_v^{'}$. Then

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'} & \mathbf{1}_{b_1} \\ \mathbf{1}_{b_1}\mathbf{1}_v^{'} - \mathbf{N}_1^{'} & -\mathbf{1}_{b_1} \\ \mathbf{N}_2^{'} - \mathbf{1}_{b_2}\mathbf{1}_v^{'} & \mathbf{0}_{b_2} \\ \mathbf{1}_{b_2}\mathbf{1}_v^{'} - \mathbf{N}_2^{'} & \mathbf{0}_{b_2} \end{bmatrix}. \qquad (4.1)$$

In this design we have $p = v+1$, $n_1 = b_1$, $n_2 = b_2$. Thus, each of first v columns of $\mathbf{X}$ contains $b_1 + b_2 - \rho_{11} - \rho_{12}$ elements equal to -1, $2(\rho_{11} + \rho_{12})$

elements equal to 0 and $b_1 + b_2 - \rho_{11} - \rho_{12}$ elements equal to 1. The last column of $\mathbf{X}$ contains b_1 elements equal to -1, $2b_2$ elements equal to 0 and b_1 elements equal to 1. Clearly, the form of such design implies that the j th object is weighted $2(b_1 + b_2 - \rho_{11} - \rho_{12})$ times, $j = 1,2,...,v$, and the $(v+1)$th object is weighted $2b_1$ times in the $2(b_1 + b_2)$ weighing operations.

Theorem 4.1. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(-1,0,1)$ given by (4.1) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is optimal for estimation unknown measurements of objects if and only if

$$b_1 - 2r_1 + \lambda_1 + b_2 - 2r_2 + \lambda_2 = 0, \tag{4.2}$$

$$b_1 = r_1 \quad \text{and} \tag{4.3}$$

$$b_2 = \rho_{11} + \rho_{12}. \tag{4.4}$$

Proof. For the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(-1,0,1)$ given by (4.1) we have

$$\mathbf{X}_1'\mathbf{1}_{b_1} = (r_1 - b_1)\mathbf{1}_v, \tag{4.5}$$

$$\mathbf{X}_1'\mathbf{X}_1 + \mathbf{X}_2'\mathbf{X}_2 = a_1\mathbf{I}_v + a_2\mathbf{1}_v\mathbf{1}_v', \tag{4.6}$$

where $a_1 = r_1 + 2\rho_{21} - \lambda_1 + r_2 + 2\rho_{22} - \lambda_2$, $a_2 = b_1 - 2r_1 + \lambda_1 + b_2 - 2r_2 + \lambda_2$. According to Theorem 2.1, the chemical balance weighing design with $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(-1,0,1)$ given by (4.1) is optimal if the conditions (2.2) and (2.3) are fulfilled. The condition (4.5) is true if and only if $b_1 = r_1$. From (2.3) and the equation (4.6) we get $a_1 = 0$ and $b_2 = \rho_{11} + \rho_{12}$. Hence the thesis.

If the chemical balance weighing design given by matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(-1,0,1)$ of the form (4.1) is optimal then $\mathrm{Var}(\hat{w}_j) = \dfrac{\sigma^2 g(1-\rho)}{2b_1}$, $j = 1,2,..., p$.

Now, let $\mathbf{N}_1$ be the incidence matrix of ternary balanced block design with the parameters v, b_1, r_1, k_1, λ_1, ρ_{11}, ρ_{21} and $\mathbf{N}_2$ be the incidence matrix of balanced incomplete block design with the parameters v, b_2, r_2, k_2, λ_2. Now, we define matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1), where $\mathbf{X}_1 = \mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'}$ and $\mathbf{X}_2 = 2\mathbf{N}_2^{'} - \mathbf{1}_{b_2}\mathbf{1}_v^{'}$. Thus

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'} & \mathbf{1}_{b_1} \\ \mathbf{1}_{b_1}\mathbf{1}_v^{'} - \mathbf{N}_1^{'} & -\mathbf{1}_{b_1} \\ 2\mathbf{N}_2^{'} - \mathbf{1}_{b_2}\mathbf{1}_v^{'} & \mathbf{0}_{b_2} \\ \mathbf{1}_{b_2}\mathbf{1}_v^{'} - 2\mathbf{N}_2^{'} & \mathbf{0}_{b_2} \end{bmatrix}. \tag{4.7}$$

In such a design we determine unknown measurements of $p = v+1$ objects. Each object is weighted $m = 2(b_1 + b_2 - \rho_{11}) = 2b_1$ times in $n = 2(b_1 + b_2)$ measurements operations.

Theorem 4.2. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by (4.7) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is optimal for estimation of unknown measurements of objects if and only if

$$b_1 - 2r_1 + \lambda_1 + b_2 - 4(r_2 - \lambda_2) = 0, \tag{4.8}$$

$$b_1 = r_1 \quad \text{and} \tag{4.9}$$

$$b_2 = \rho_{11}. \tag{4.10}$$

Proof. Proof of this theorem is similar to proof of Theorem 4.1.

In particular case, when $\mathbf{N}_2 = \mathbf{1}_v\mathbf{1}_b^{'}$ then the design is called randomize block design and we have the following corollary.

Corollary 4.1. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ of the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'} & \mathbf{1}_{b_1} \\ \mathbf{1}_{b_1}\mathbf{1}_v^{'} - \mathbf{N}_1^{'} & -\mathbf{1}_{b_1} \\ \mathbf{1}_{b_2}\mathbf{1}_v^{'} & \mathbf{0}_{b_2} \\ -\mathbf{1}_{b_2}\mathbf{1}_v^{'} & \mathbf{0}_{b_2} \end{bmatrix} \tag{4.11}$$

with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1), is optimal for estimation of unknown measurements of objects if and only if

$$b_1 - 2r_1 + \lambda_1 + b_2 = 0 \tag{4.12}$$

and the conditions (4.9) and (4.10) are true.

Now, let $\mathbf{N}_1$ be the incidence matrix of ternary balanced block design with the parameters v, b_1, r_1, k_1, λ_1, ρ_{11}, ρ_{21} and $\mathbf{N}_2^*$ be the incidence matrix of balanced bipartite weighing design with the parameters v, b_2, r_2, k_{12}, k_{22}, λ_{12}, λ_{22}. Using the matrix $\mathbf{N}_2^*$ we form the matrix $\mathbf{N}_2$ by replacing k_{12} elements equal to +1 of each column which correspond to the elements belonging to the first subblock by -1. Then each column of the matrix $\mathbf{N}_2$ contains k_{12} elements equal to -1, $v - k_{12} - k_{22}$ elements equal to 0 and k_{22} elements equal to 1. Now, we define the matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form (2.1), where $\mathbf{X}_1 = \mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'}$ and $\mathbf{X}_2 = \mathbf{N}_2^{'}$. Hence

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1^{'} - \mathbf{1}_{b_1}\mathbf{1}_v^{'} & \mathbf{1}_{b_1} \\ \mathbf{1}_{b_1}\mathbf{1}_v^{'} - \mathbf{N}_1^{'} & -\mathbf{1}_{b_1} \\ \mathbf{N}_2^{'} & \mathbf{0}_{b_2} \\ -\mathbf{N}_2^{'} & \mathbf{0}_{b_2} \end{bmatrix}. \tag{4.13}$$

In such design we determine unknown measurements of $p = v + 1$ objects. Each object is weighted $m = 2(b_1 - \rho_{11} + r_2) = 2b_1$ times in $n = 2(b_1 + b_2)$ measurements operations.

Theorem 4.3. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by (4.13) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1), is optimal for estimation of unknown measurements of objects if and only if

$$b_1 - 2r_1 + \lambda_1 + \lambda_{22} - \lambda_{12} = 0\,, \tag{4.14}$$

$$b_1 = r_1 \quad \text{and} \tag{4.15}$$

$$r_2 = \rho_{11}\,. \tag{4.16}$$

Proof. Proof of Theorem 4.3 is similar to proof of Theorem 4.1.

5. The block designs leading to optimal weighing designs

From Theorem 4.1 follows that if parameters of two ternary balanced block designs satisfy the conditions (4.2)-(4.4) then a chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ of the form (4.1) is optimal. Under these conditions we formulate a theorem following from the papers of Swamy (1982), Billington and Robinson (1983), Billington (1984) and Ceranka and Graczyk (2004).

Theorem 5.1. The existence of two ternary balanced block designs with the parameters

(i) $v = 5$, $b_1 = 4(s+2)$, $r_1 = 4(s+2)$, $k_1 = 5$, $\lambda_1 = 4s+7$, $\rho_{11} = 4(s+1)$, $\rho_{21} = 2$ and $v = 5$, $b_2 = 5(s+2)$, $r_2 = 3(s+2)$, $k_2 = 3$, $\lambda_2 = s+3$, $\rho_{12} = s+6$, $\rho_{22} = s$, $s = 1,2,\ldots$,

(ii) $v = 5$, $b_1 = 4(s+4)$, $r_1 = 4(s+4)$, $k_1 = 5$, $\lambda_1 = 2(2s+7)$, $\rho_{11} = 4(s+2)$, $\rho_{21} = 4$ and $v = 5$, $b_2 = 5(s+4)$, $r_2 = 3(s+4)$, $k_2 = 3$, $\lambda_2 = s+6$, $\rho_{12} = s+12$, $\rho_{22} = s$, $s = 1,2,\ldots$,

(iii) $v = 6$, $b_1 = 3(s+5)$, $r_1 = 3(s+5)$, $k_1 = 6$, $\lambda_1 = 3s+13$, $\rho_{11} = 3s+5$, $\rho_{21} = 5$ and $v = 6$, $b_2 = 2(s+5)$, $r_2 = s+5$, $k_2 = 3$, $\lambda_2 = 2$, $\rho_{12} = 5-s$, $\rho_{22} = 4$, $s = 1,2,3,4$,

(iv) $v = 7$, $b_1 = 27$, $r_1 = 27$, $k_1 = 7$, $\lambda_1 = 25$, $\rho_{11} = 15$, $\rho_{21} = 6$ and $v = 7$, $b_2 = 21$, $r_2 = 12$, $k_2 = 4$, $\lambda_2 = 5$, $\rho_{12} = 6$, $\rho_{22} = 3$,

(v) $v=9$, $b_1=3(s+14)$, $r_1=3(s+14)$, $k_1=9$, $\lambda_1=3s+11$, $\rho_{11}=3s+4$, $\rho_{21}=4$ and $v=9$, $b_2=3(s+4)$, $r_2=2(s+4)$, $k_2=6$, $\lambda_2=s+5$, $\rho_{12}=8$, $\rho_{22}=s$, $s=1,2,\ldots$,

(vi) $v=11$, $b_1=16$, $r_1=16$, $k_1=11$, $\lambda_1=15$, $\rho_{11}=6$, $\rho_{21}=5$ and $v=11$, $b_2=11$, $r_2=7$, $k_2=7$, $\lambda_2=4$, $\rho_{12}=5$, $\rho_{22}=1$,

(vii) $v=15$, $b_1=5(s+4)$, $r_1=5(s+4)$, $k_1=5$, $\lambda_1=5s+19$, $\rho_{11}=5s+6$, $\rho_{21}=7$ and $v=15$, $b_2=3(s+4)$, $r_2=2(s+4)$, $k_2=10$, $\lambda_2=s+5$, $\rho_{12}=6-2s$, $\rho_{22}=2s+1$, $s=1,2$,

implies the existence of the optimal chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ given by (4.1) and with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1).

Proof. It is easy to prove that the parameters of ternary balanced block design satisfy the conditions (4.2)-(4.4).

From Theorem 4.2 follows that if parameters of ternary balanced block designs and balanced incomplete block designs satisfy the conditions (4.8)-(4.10) then a chemical balance weighing design with the design matrix $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ of the form (4.7) is optimal. Parameters satisfying the conditions (4.8)-(4.10) are given in the following Theorem.

Theorem 5.2. The existence of ternary balanced block designs and balanced incomplete block designs with the parameters

(i) $v=7$, $b_1=54$, $r_1=54$, $k_1=7$, $\lambda_1=52$, $\rho_{11}=42$, $\rho_{21}=6$ and $v=7$, $b_2=42$, $r_2=12$, $k_2=2$, $\lambda_2=2$,

(ii) $v=10$, $b_1=48$, $r_1=48$, $k_1=10$, $\lambda_1=46$, $\rho_{11}=30$, $\rho_{21}=9$ and $v=10$, $b_2=30$, $r_2=9$, $k_2=3$, $\lambda_2=2$,

(iii) $v=13$, $b_1=50$, $r_1=50$, $k_1=13$, $\lambda_1=48$, $\rho_{11}=26$, $\rho_{21}=12$ and $v=13$, $b_2=26$, $r_2=8$, $k_2=4$, $\lambda_2=2$,

(iv) $v=15$, $b_1=70$, $r_1=70$, $k_1=15$, $\lambda_1=68$, $\rho_{11}=42$, $\rho_{21}=14$ and $v=15$, $b_2=42$, $r_2=14$, $k_2=5$, $\lambda_2=4$,

(v) $v=21$, $b_1=70$, $r_1=70$, $k_1=21$, $\lambda_1=68$, $\rho_{11}=30$, $\rho_{21}=20$ and $v=21$, $b_2=30$, $r_2=10$, $k_2=7$, $\lambda_2=2$,

implies the existence of the optimal chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ given by (4.7) and with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1).

Proof. One can easy check that the parameters of ternary balanced block design satisfy the conditions (4.8)-(4.10).

In particular case when design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ is given by (4.11) we have the following Theorem.

Theorem 5.3. The existence of ternary balanced block designs and randomized block design with the parameters

(i) $v=12$, $b_1=18$, $r_1=15$, $k_1=10$, $\lambda_1=11$, $\rho_{11}=1$, $\rho_{21}=7$ and $v=12$, $b_2=1$,

(ii) $v=s$, $b_1=s$, $r_1=s$, $k_1=s$, $\lambda_1=s-1$, $\rho_{11}=1$, $\rho_{21}=0{,}5(s-1)$ and $v=s$, $b_2=1$, $s=5, 9, 11, 15$,

(iii) $v=4s+3$, $b_1=4s+3$, $r_1=4s+3$, $k_1=4s+3$, $\lambda_1=2(s+1)$, $\rho_{11}=1$, $\rho_{21}=2s+1$ and $v=4s+3$, $b_2=1$, $s=1,2,\ldots$,

(iv) $v=2s+1$, $b_1=2(2s+1)$, $r_1=2(2s+1)$, $k_1=2s+1$, $\lambda_1=4s$, $\rho_{11}=2$, $\rho_{21}=2s$ and $v=2s+1$, $b_2=2$, $s=1,2,\ldots$,

(v) $v=2s$, $b_1=4s$, $r_1=4s$, $k_1=2s$, $\lambda_1=2(2s-1)$, $\rho_{11}=2$, $\rho_{21}=2s-1$ and $v=2s$, $b_2=2$, $s=1,2,\ldots$,

implies the existence of the optimal chemical balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ given by (4.7) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1).

Proof. The parameters of ternary balanced block design satisfy the conditions (4.9), (4.10) and (4.12).

Now, we give the parameters of ternary balanced block designs and balanced bipartite block designs based on which we form the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ of the optimal chemical balance weighing design in the form (4.13).

Theorem 5.4. The existence of ternary balanced block designs and balanced bipartite block designs with the parameters

(i) $v = 2s+1$, $b_1 = 9s$, $r_1 = 9s$, $k_1 = 2s+1$, $\lambda_1 = 9s-1$, $\rho_{11} = 7s$, $\rho_{21} = s$ and $v = 2s+1$, $b_2 = s(2s+1)$, $r_2 = 7s$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$, $s = 1,2,\ldots$,

(ii) $v = 4s+1$, $b_1 = 16s$, $r_1 = 16s$, $k_1 = 4s+1$, $\lambda_1 = 2(8s-1)$, $\rho_{11} = 8s$, $\rho_{21} = 4s$ and $v = 4s+1$, $b_2 = s(4s+1)$, $r_2 = 8s$, $k_{12} = 2$, $k_{22} = 6$, $\lambda_{12} = 6$, $\lambda_{22} = 8$, $s = 1,2,\ldots$,

(iii) $v = 2s$, $b_1 = 9(2s-1)$, $r_1 = 9(2s-1)$, $k_1 = 2s$, $\lambda_1 = 18s-11$, $\rho_{11} = 7(2s-1)$, $\rho_{21} = 2s-1$ and $v = 2s$, $b_2 = 2s(2s-1)$, $r_2 = 7(2s-1)$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 20$, $\lambda_{22} = 22$, $s = 2,3,\ldots$,

(iv) $v = 4s+1$, $b_1 = 9s$, $r_1 = 9s$, $k_1 = 4s+1$, $\lambda_1 = 9s-1$, $\rho_{11} = 5s$, $\rho_{21} = 2s$ and $v = 4s+1$, $b_2 = s(4s+1)$, $r_2 = 5s$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$, $s = 1,2$,

(v) $v = 11$, $b_1 = 16$, $r_1 = 16$, $k_1 = 11$, $\lambda_1 = 15$, $\rho_{11} = 6$, $\rho_{21} = 5$ and $v = 11$, $b_2 = 11$, $r_2 = 6$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 1$, $\lambda_{22} = 2$

implies the existence of the optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by (4.7) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1).

Proof. One can prove that the parameters of ternary balanced block design satisfy the conditions (4.14)-(4.16).

Let us consider any ρ_h, $\rho_h \in \left(\frac{-1}{n-1}, 0 \right]$, $h = 1,2$, $\rho_1 \neq \rho_2$. It is worth to point out that the design $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(-1,0,1)$ satisfying Theorems 4.1-4.3 is optimal for the estimation of unknown measurements of objects in the sense of attaining minimal variance of the estimator of unknown measurements of objects for the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1.1) for ρ_1 and ρ_2. Simultaneously, the lower bound of variance given in (1.2) is not the same for different numbers of ρ. For a dipper discussion of robustness optimal design we refer the reader to Masaro and Wong (2008).

6. Example

Let us consider an experiment in which we determine unknown measurements of $p = 6$ objects using $n = 24$ measurement operations under the assumption that each object is weighted at least $m = 20$ times. To construct the design matrix $\mathbf{X} \in \mathbf{\Phi}_{24 \times 6}(-1,0,1)$ of the optimal chemical balance weighing design we use the incidence matrix of ternary balanced block design with parameters $v = 5, b = 10, r = 10, k = 5, \lambda = 8$, $\rho_1 = 2$, $\rho_2 = 4$, given by the incidence matrix

$$\mathbf{N} = \begin{bmatrix} 2 & 2 & 1 & 0 & 2 & 2 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 2 & 2 & 0 & 2 & 1 & 0 & 0 & 2 \\ 0 & 2 & 2 & 1 & 0 & 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 1 & 2 & 2 & 1 & 2 \end{bmatrix}.$$

Then we built the design matrix $\mathbf{X} \in \mathbf{\Phi}_{24 \times 6}(-1,0,1)$ of optimal chemical balance weighing design in the form (4.11) and we have

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{1}_{10} \\ -\mathbf{X}_1 & -\mathbf{1}_{10} \\ \mathbf{1}_2\mathbf{1}_5 & \mathbf{0}_2 \\ -\mathbf{1}_2\mathbf{1}_5 & \mathbf{0}_2 \end{bmatrix}, \text{ where } \mathbf{X}_1 = \begin{bmatrix} 1 & 1 & 0 & -1 & -1 \\ 1 & 0 & -1 & 1 & -1 \\ 0 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 0 & -1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 0 \\ -1 & 1 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 & 1 \end{bmatrix}. \text{ For the}$$

$\mathbf{X} \in \mathbf{\Phi}_{24\times 6}(-1,0,1)$ design with covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1.1) we have $\mathrm{Var}(\hat{w}_j) = \sigma^2 \frac{g(1-\rho)}{20}, \quad j = 1,2,...,6$.

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