Colloquium Biometricum 42
2012, 77–84

# CONTROL WHOLE PLOT TREATMENTS IN SOME SPLIT-SPLIT-PLOT EXPERIMENT DESIGNS GENERATED BY ORTHOGONALLY SUPPLEMENTED BLOCK DESIGN

**Katarzyna Ambroży, Iwona Mejza**

Department of Mathematical and Statistical Methods
Poznań University of Life Science
Wojska Polskiego 28, 60-637 Poznań, Poland
e-mail: ambrozy@up.poznan.pl, imejza@up.poznan.pl

## Summary

In the paper we consider a situation when a split-split-plot design is non-orthogonal with respect to whole plot treatments. Additionally, some of them are treated as control treatments. To generate a new treatment combination arrangement an orthogonally supplemented PEB block design with at most ($m$ + 1) – classes of efficiency is taken into account. We present also statistical properties of the resulting design and an example.

**Keywords and phrases:** control treatments, general balance, supplemented block design, split-split-plot design, stratum efficiency factors

**Classification AMS 2010:** 62K10, 62K15

## 1. Introduction

This paper is a continuation of the paper of Ambroży and Mejza (2011) in which a traditional split-split-plot (SSP) design (e.g. Gomez and Gomez, 1984) was recalled. In the previous paper mainly we discussed statistical implications of a proposed non-orthogonality of the layout when the sub-subplot treatments occurred in some supplemented partially efficiency block (PEB) design with

$(m + 1)$ – classes of efficiency. We also presented the randomization-derived mixed model of observations for complete and incomplete SSP designs and statistical properties such as orthogonal block structure and general balance. We defined so called strata in the model and stratum effectiveness of that SSP design with respect to estimable contrasts in them.

In the present paper we consider a non-orthogonal SSP design in which whole plot treatments occur in a supplemented PEB design with $(m + 1)$ – classes of efficiency while the subplot treatments and the sub-subplot treatments are in appropriate randomized complete block (RCB) designs.

The supplemented (augmented) block designs for one-factor experiments have been widely described in literature (e.g. Caliński 1971, Caliński and Ceranka 1974, Singh and Dey 1979, Puri et al. 1977, Kachlicka and Mejza 1998, Caliński and Kageyama 2003, Sections 6.3. and 10.3.3). Generally, two sets of treatments are there. Usually one set is referred to as the set of basic (test) treatments and the other - the set of supplementary (control) treatments. The major aim of such experiments is the efficient comparison of both sets of treatments and the treatments inside those sets.

This fact has been used in the construction of a new non-orthogonal layout of the SSP experiment in which there are additional whole plot treatments called control whole plot treatments.

## 2. Assumptions and notations

Let us consider a three-factor experiment in which the first factor, say $A$, has $s$ levels $A_1, A_2, \ldots, A_s$, (called also the whole plot treatments), the second factor, say $B$, has $t$ levels $B_1, B_2, \ldots, B_t$ (called the subplot treatments) and the third factor, say $C$, has $w$ levels $C_1, C_2, \ldots, C_w$ (called the sub-subplot treatments). Thus the number $v = stw$ denotes the number of all treatment combinations in the experiment.

There is assumed the experimental material can be divided into b blocks with $k_1 < s$ whole plots. Then, each whole plot is divided into $k_2 = t$ subplots with $k_3 = w$ sub-subplots. The $s$ whole plot (A) treatments are randomly allocated to the whole plots within each block, t subplot (B) treatments are randomly allocated to the subplots within each whole plot, and w (C) sub-subplot treatments are randomly allocated to the sub-subplots within each subplot. Hence, the third factor C is in a split-plot relation to the whole plot and subplot treatment combinations in the SSP design. Next in the paper we adopt the following notation: $\mathbf{1}_x$ is the x-dimensional vector of ones, $\mathbf{I}_x$ denotes $x$-dimensional unity matrix and $\mathbf{J}_x = \mathbf{1}_x\mathbf{1}'_x$.

## 3. Construction method

This method is based on Kronecker product of three designs, in which the levels of three factors ($A, B, C$) are assigned. Consider a situation when *t* subplot (*B*) treatments and *w* sub-subplot (*C*) treatments are in appropriate RCB designs whereas the *s* whole plot (*A*) treatments occur in a supplemented block design $d^*(v^* = s,\ b^*,\ k^*,\ \mathbf{r}^*)$, where the parameters $v^*$, $b^*$, $k^*$ mean numbers of the whole plot treatments, blocks, units inside each block in the subdesign $d^*$, respectively and $\mathbf{r}^*$ denotes a vector of replicates of the all whole plot treatments.

We also assume the whole plot (*A*) treatments consist of two groups: $s = s_1 + s_2$, where $s_1$ test (basic) $A$ treatments are allocated in a subdesign $\tilde{d}_1$ which is a partially efficiency balanced (PEB) design with at most *m* efficiency classes (cf. Puri et al. 1977, Kageyama and Puri 1985, Caliński and Kageyama 2000, Definition 4.3.1.) while $s_2$ additional (control) $A$ treatments – in a subdesign $\tilde{d}_2$ represented by an orthogonal block design (cf. Caliński and Kageyama, 2000, Definitions 2.2.7-2.2.8).

Let $\tilde{\mathbf{N}}_1$ be the $s_1 \times b_1$ incidence matrix of the subdesign $\tilde{d}_1$ with parameters: $s_1$, $b_1$, $\tilde{k}_1$, $\mathbf{r}_{s_1} = [r_1, r_2, ..., r_{s_1}]'$ $\varepsilon_j$, $\rho_j$ $(\sum_{j=1}^{m} \rho_j = s_1 - 1)$, which define number of treatments, number of blocks, size of blocks, vector of treatment replicates, eigenvalues and their multiplicities of so-called **C**- matrix of the subdesign $\tilde{d}_1$, respectively (see the example in Chapter 4). Then (cf. Puri and Nigam, 1977, Nigam and Puri, 1982, Caliński and Kageyama, 2003, e.g. Theorems 6.3.1. and 10.3.3.)

$$\mathbf{N}_{d*} = \begin{bmatrix} \tilde{\mathbf{N}}_1 \\ \mathbf{r}_{s_2}(\mathbf{k}^*)'/n* \end{bmatrix}, \tag{3.1}$$

is the incidence matrix of the PEB design with at most (m + 1)-classes of efficiency with parameters:

$$v^* = s = s_1 + s_2,\quad b^* = b_1,\quad \mathbf{k}^* = k^*\mathbf{1}_{b_1} = n^*\tilde{k}_1\mathbf{1}_{b_1}/\tilde{n}_1,$$
$$\mathbf{r}^* = \mathbf{N}_{d^*}\mathbf{1}_s = [\mathbf{r}'_{s_1}, \mathbf{r}'_{s_2}]' = [r_1, r_2, ..., r_{s_1}, r_{s_1+1}, ..., r_s]',$$

$$\varepsilon_0^* = 1,\quad \rho_0^* = s_2,\ \varepsilon_j^* = 1 - (\tilde{n}_1/n^*)(1 - \varepsilon_j),\quad \rho_j^* = \rho_j,\quad j = 1, 2, \ldots, m \tag{3.2}$$

where $\tilde{n}_1$ and $n^*$ denote numbers of observations in the designs $\tilde{d}_1$ and $d^*$, respectively.

Let $\mathbf{N}_1 = \mathbf{N}_{d^*} \otimes \mathbf{1}_{tw}$ be the $v \times b$ incidence matrix of the considered SSP design with parameters: $v = (s_1 + s_2)tw$, $b = b^*$, $k = k^* tw$, $\mathbf{r} = \mathbf{r}^* \otimes \mathbf{1}_{tw}$, $n = bk^* tw$, where $\mathbf{N}_{d^*}$ is given in (3.1). This incidence matrix $\mathbf{N}_1$ with respect to blocks plays an important role in constructing the new SSP design. The applied construction leads to proper (cf. Caliński and Kageyama 2000, Definition 2.2.2) and non-equireplicated experiment SSP design (cf. Caliński and Kageyama 2000, Definition 2.2.3).

As mentioned by Ambroży and Mejza (2011), statistical properties of the SSP designs are related mainly to algebraic properties of stratum information matrices $\mathbf{A}_f$, $f$ = 0, 1, …, 4. In the present case, forms of these matrices are given in (3.3).

Assuming that $(\mathbf{r}^*)^\delta = \mathrm{diag}(r_1, r_2, ..., r_s)$, where $\mathbf{r}^*$ is in (3.2), we have

$$
\begin{aligned}
\mathbf{A}_0 &= \frac{1}{bk^*tw}\mathbf{r}^*(\mathbf{r}^*)' \otimes \mathbf{J}_{tw} \\
\mathbf{A}_1 &= \frac{1}{k^*tw}\mathbf{N}_{d^*}\mathbf{N}'_{d^*} \otimes \mathbf{J}_{tw} - \frac{1}{bk^*tw}\mathbf{r}^*(\mathbf{r}^*)' \otimes \mathbf{J}_{tw}, \\
\mathbf{A}_2 &= \frac{1}{tw}(\mathbf{r}^*)^\delta \otimes \mathbf{J}_{tw} - \frac{1}{k^*tw}\mathbf{N}_{d^*}\mathbf{N}'_{d^*} \otimes \mathbf{J}_{tw}, \\
\mathbf{A}_3 &= \frac{1}{w}(\mathbf{r}^*)^\delta \otimes \mathbf{I}_t \otimes \mathbf{J}_w - \frac{1}{tw}(\mathbf{r}^*)^\delta \otimes \mathbf{J}_{tw}, \\
\mathbf{A}_4 &= (\mathbf{r}^*)^\delta \otimes \mathbf{J}_{tw} - \frac{1}{w}(\mathbf{r}^*)^\delta \otimes \mathbf{I}_t \otimes \mathbf{J}_w .
\end{aligned}
\tag{3.3}
$$

One can check that resulting SSP design is generally balanced. It follows from the fact the matrices (3.3) commute with respect to $\mathbf{r}^{-\delta} = (\mathbf{r}^*)^{-\delta} \otimes \mathbf{I}_{tw}$ (e.g. Mejza, 1992, Ambroży and Mejza, 2011), where $(\mathbf{r}^*)^{-\delta} = \mathrm{diag}(1/r_1, ..., 1/r_s)$. This means that these matrices have a common set of eigenvectors corresponding to some eigenvalues with respect to $\mathbf{r}^\delta$. It allows to define a common set of contrasts and corresponding to them stratum efficiency factors (cf. Mejza, 1997a, 1997b).

**Table 1**. Stratum efficiency factors of the considered non-orthogonal SSP design

| **Types of contrasts** | **df** | **Strata** | | | |
|---|---|---|---|---|---|
| | | **1** | **2** | **3** | **4** |
| $A^T$ | $\left.\begin{matrix}\rho_1^*\\ \cdots\\ \rho_m^*\end{matrix}\right\} = s_1 - 1$ | $1-\varepsilon_1^*$ <br> $\cdots$ <br> $1-\varepsilon_m^*$ | $\varepsilon_1^*$ <br> $\cdots$ <br> $\varepsilon_m^*$ | | |
| $A^C$ | $s_2 - 1$ | | $\varepsilon_0^* = 1$ | | |
| $A^T$ *vs.* $A^C$ | 1 | | $\varepsilon_0^* = 1$ | | |
| $B$ | $t-1$ | | | 1 | |
| $C$ | $w-1$ | | | | 1 |
| $B \times C$ | $(t-1)(w-1)$ | | | | 1 |
| $A^T \times B$ | $\left.\begin{matrix}\rho_1^*(t-1)\\ \cdots\\ \rho_m^*(t-1)\end{matrix}\right\} = (s_1 - 1)(t-1))$ | | | $\varepsilon_0^* = 1$ | |
| $A^C \times B$ | $(s_2 - 1)(t-1)$ | | | $\varepsilon_0^* = 1$ | |
| ($A^T$ *vs.* $A^C$) $\times B$ | $t-1$ | | | $\varepsilon_0^* = 1$ | |
| $A^T \times C$ | $\left.\begin{matrix}\rho_1^*(w-1)\\ \cdots\\ \rho_m^*(w-1)\end{matrix}\right\} = (s_1 - 1)(w-1)$ | | | | $\varepsilon_0^* = 1$ |
| $A^C \times C$ | $(s_2 - 1)(w-1)$ | | | | $\varepsilon_0^* = 1$ |
| ($A^T$ *vs.* $A^C$) $\times C$ | $w-1$ | | | | $\varepsilon_0^* = 1$ |
| $A^T \times B \times C$ | $\left.\begin{matrix}\rho_1^*(t-1)(w-1)\\ \cdots\\ \rho_m^*(t-1)(w-1)\end{matrix}\right\} = (s_1 - 1)(t-1)(w-1)$ | | | | $\varepsilon_0^* = 1$ |
| $A^C \times B \times C$ | $(s_2 - 1)(t-1)(w-1)$ | | | | $\varepsilon_0^* = 1$ |
| ($A^T$ *vs.* $A^C$) $\times B \times C$ | $(t-1)(w-1)$ | | | | $\varepsilon_0^* = 1$ |

df (*degrees of freedom*) – numbers of the particular types of the contrasts estimable in the strata; 1– the inter-block stratum, 2 – the inter-whole plot stratum, 3 – the inter-subplot stratum, 4 – the inter-sub-subplot stratum

In the present paper, we consider the following types of the contrasts: among main effects of the whole plot treatments including: test A treatments ($A^T$) and additional (control) A treatments ($A^C$), between the test group and the control group of A treatments ($A^T$ vs. $A^C$), then among main effects of the subplot (B) treatments, among main effects of the sub-subplot (C) treatments, and other interaction contrasts as in table 1. Analyzing algebraic properties of the matrices (3.3) we obtain information about estimability of the contrasts in the strata and their stratum efficiency factors $\varepsilon_{fh}$, $f = 0, 1, \ldots, 4$; $h = 1, 2, \ldots, v$; $h < v$. In the table 1, they are expressed by the eigenvalues $\varepsilon_j^*$, $j = 0, 1, \ldots, m$, given in (3.2), according to the construction method.

## 4. Some remarks and example

From Table 1, it follows that we lose less information in the incomplete SSP design with respect to the whole plot (*A*) treatments than in other cases (e.g. Ambroży and Mejza, 2011).

We can notice that only basic contrasts among main effects of the test *A* treatments ($A^T$) are estimated with partial efficiency in two different strata: in the inter-block stratum and the inter-whole plot stratum (*m* classes of efficiency). All other contrasts are estimated with full efficiency (= 1). It means that information about these contrasts is contained in only one, corresponding to the type of a contrast, stratum. It follows from the construction method (the subplot treatments and the sub-subplot treatments are in RCB subdesigns), from the nature of the SSP design (a nested system of units) and from statistical properties of the generating design (an orthogonal supplementation).

To reduce the number of *m* efficiency classes of the subdesign $\tilde{d}_1$ (and thus the generated SSP design) we chose the PEB design with *m* – efficiency classes from the class of the PBIB designs (e.g. Clatworthy (1973)). We assumed that $s_1$ test *A* treatments can be divided into $l_1$ groups with $l_2$ different test *A* treatments, so $s_1 = l_1 l_2$. Let the test *A* treatments occur in a regular group divisible partially incomplete block design with two efficiency classes (R-GDPBIB(2) design) denoted by $\tilde{d}_1$ ($s_1 = l_1 l_2$, $b_1$, $\tilde{k}_1$, $\mathbf{r}_{s_1} = \tilde{r}\mathbf{1}_{s_1}$, $\lambda_1$, $\lambda_2$), where $b_1$, $\tilde{k}_1$ are defined in the third chapter of the present paper while $\tilde{r}$, $\lambda_1$, $\lambda_2$ mean a number of replicates of the test *A* treatments and numbers of meetings in blocks of pairs of these treatments belonging to the same group or different groups, respectively. The statistical properties of the designs considered are described in terms of eigenvalues $\mu_0 = \tilde{r}\tilde{k}_1$ with multiplicity 1, $\mu_1 = \tilde{r} - \lambda_1$ with $\rho_1 = l_1(l_2 - 1)$ and $\mu_2 = \tilde{r}\tilde{k}_1 - s_1\lambda_2$ with $\rho_2 = l_1 - 1$ of the association matrix $\tilde{\mathbf{N}}_1\tilde{\mathbf{N}}_1'$ of the GDPBIB(2) design, where $\tilde{\mathbf{N}}_1$ is defined in Chapter 3 and $\sum_{j=0}^{2}\rho_j = s_1$. Let $\tilde{\mathbf{C}}_1 = \tilde{r}\mathbf{I} - \tilde{k}_1\tilde{\mathbf{N}}_1\tilde{\mathbf{N}}_1'$ be **C**-information matrix for the test $A$ treatments and $\varepsilon_j = 1 - \mu_j / \tilde{r}\tilde{k}_1$, denote eigenvalues of this matrix with respect to $\tilde{r}\mathbf{I}$ with multiplicities $\rho_j$, $j$ = 0, 1, 2, where $\sum_{j=0}^{2}\rho_j = s_1$. Let us note that no contrast is connected with the eigenvalue $\varepsilon_0 = 0$, so we will omit it in further considerations. Using above considerations we can write eigenvalues (3.2) of an C-information matrix for all $A$ treatments as follows:

$$\varepsilon_0^* = 1,\ \rho_0^* = s_2,\ \varepsilon_j^* = 1 - (\tilde{n}_1 / n^*)\frac{\mu_j}{\tilde{r}\tilde{k}_1},\ \rho_j^* = \rho_j,\ j = 1, 2. \quad (4.1)$$

It is convenient to introduce an abbreviation to describe the property of balance of the considered SSP design. Let $M_f\{q, \alpha\}$ denote the property that $q$ contrasts between main effects of factor $M$ (or interaction contrasts) are estimated in the $f$-th stratum with efficiency factor $\alpha$. In other words, we say that design is $\mathrm{M}_f\{q, \alpha\}$-balanced. In particular, if $\alpha = 1$ that design is $\mathrm{M}_f\{q, 1\}$-orthogonal.

In the example we can say the considered non-orthogonal SSP design with the number of the treatment combinations $v = stw$, where $s = s_1 + s_2$, generated by R-GDPBIB(2) design with the number of the test $A$ treatments $s_1 = l_1 l_2$ is:

$A_1\{l_1(l_2 - 1), 1 - \varepsilon_1^*\}$-balanced, $A_1\{l_1 - 1, 1 - \varepsilon_2^*\}$-balanced,

$A_2\{l_1(l_2 - 1), \varepsilon_1^*\}$-balanced, $A_2\{l_1 - 1, \varepsilon_2^*\}$-balanced,

where $\varepsilon_1^*$ and $\varepsilon_2^*$ are given in (4.1) and with respect to other contrasts the considered SSP design is always:

$B_3\{t - 1, 1\}$-orthogonal,

$(A \times B)_3\{(s-1)(t-1), 1\}$-orthogonal,

$C_4(w - 1, 1)$-orthogonal,

$(A\times C)_4\{(s-1)(w-1),\ 1\}$-orthogonal,
$(B\times C)_4\{(t-1)(w-1),\ 1\}$-orthogonal, and
$(A\times B\times C)_4\{(s-1)(t-1)(w-1),\ 1\}$-orthogonal (cf. Table 1).

**Acknowledgement**

The paper was partially supported by project No 1116/B/P01/2011/40 (NCN-Poland).

## References

Ambroży K., Mejza I. (2011). Statistical properties of some supplemented split-split-plot design. *Colloquium Biometricum* 41, 165–174.

Caliński T. (1971). On some desirable patterns in block designs. *Biometrics* 27, 275–292.

Caliński T., Ceranka B. (1974). Supplemented block designs. *Biom. J.* 16, 299–305.

Caliński, T., Kageyama, S. (2000). *Block Designs. A Randomization Approach*, Volume I. *Analysis*. Lecture Notes in Statistics 150, Springer–Verlag, New York.

Caliński, T., Kageyama, S. (2003). *Block Designs. A Randomization Approach*, Volume II. *Design*. Lecture Notes in Statistics 170, Springer –Verlag, New York.

Clatworthy W.H. (1973). *Tables of two associate class partially balanced designs*. NBS App. Math. Ser. 63, Department of Commerce.

Gomez K.A, Gomez A.A. (1984). *Statistical procedures for agricultural research*. Wiley, New York.

Kachlicka D., Mejza I. (1998). Supplemented block designs with split units. *Colloquium Biometryczne* 28, 77–90.

Kageyama S., Puri P.D. (1985). A new special class of PEB designs. *Commun. Statist. Part A–Theor. Meth*. 14, 1731–1744.

Mejza I. (1997a). Doświadczenia trójczynnikowe w niekompletnych układach o poletkach podwójnie rozszczepionych. *Colloquium Biometryczne* 27, 127–138.

Mejza I. (1997b). Incomplete split-split-plot designs. *Abstracts of the 51st Session of the International Statistical Institute ISI'97*, Contributed Papers, Book 1, 221–222.

Mejza S. (1992). On some aspects of general balance in designed experiments. *Statistica* 52, 263–278.

Nigam A.K., Puri P.D. (1982). On partially efficiency balanced designs–II. *Commun. Statist.- Theor. Meth*. 11(24), 2817–2830.

Puri P.D., Nigam A.K. (1977). Partially efficiency balanced designs. *Commun. Statist.- Theor. Meth.* 6, 753–771.

Puri P.D., Nigam A.K., Narain P. (1977). Supplemented block designs. *Sankhyā*, Ser. B, 39, 189–195.

Singh M., Dey A. (1979). On analysis of some augmented block designs. *Biom. J.* 21, 87–92.