Colloquium Biometricum 43
2013, 13–19

ON THE CONSTRUCTION OF REGULAR A-OPTIMAL SPRING BALANCE WEIGHING DESIGNS

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Summary

In this paper we study the problem of the estimation of individual measurements of objects in spring balance weighing design under the assumption that errors are uncorrelated and have different variances. The incidence matrices of the balanced incomplete block designs are used for new construction of the regular A-optimal spring balance weighing design. Theoretical research is illustrated by an example.

Keywords and phrases: A-optimal spring balance weighing design, balanced incomplete block design

Classification AMS 2010: 62K05, 62K10

1. Introduction

The optimality of designs plays a main role in the theory of the experimental designs. In many papers concerning the optimality, the weighing designs are considered. In a spring balance, there is only one pan and any number of objects can be placed on the pan. Then the pointer provides a reading which represents the total weight of the objects on the pan.

Nowadays, the spring balance weighing design is the name for the experimental design connected not only with a spring balance, but with any experiment in that the results we can describe as the linear combination of unknown measurements of objects with coefficients of this combination equal to 1 or 0. In fact, the weighing designs are applicable to a great variety of problems of measurements, not only for weights, but of length, voltages, resistance and concentrations of chemical in mixture, analyzing the lines of legume.

The main idea is to determine unknown measurements of p objects in n weighing operations. We shall make two standing assumptions on the maps under consideration. Recorded observations are independent and there are not systematic errors. The second basic assumption is that the errors have different variances. Of course, the experimenter wants to choose a weighing design that is optimal with respect to some criterion. In literature, several criteria are often expressed in terms of the information matrix. One of them is A-optimality, in that we study the trace of the inverse of information matrix.

2. The linear model

Suppose, there are p objects of unknown measurements $w_1, w_2, ..., w_p$, respectively, and we wish to estimate them employing n measurement operations using a spring balance. Let $y_1, y_2, ..., y_n$ denote recorded observations in these n operations. It is assumed that the observations follow the model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e} \tag{2.1}$$

where $\mathbf{y} = (y_1, y_2, ..., y_n)'$ is an $n \times 1$ random vector of the observations. The design matrix $\mathbf{X} = (x_{ij})$, usually called weighing matrix, belongs to the class $\mathbf{\Psi}_{n\times p}\{0, 1\}$-which denotes the class of $n \times p$ matrices of known elements $x_{ij} = 0$ or 1 as in the i th weighing operation the j th object is not placed on the pan or is placed. A $p \times 1$ vector $\mathbf{w} = (w_1, w_2, ..., w_p)'$ contains unknown weights of objects and $\mathbf{e}$ is an $n \times 1$ random vector of errors. We assume that in the model (2.1) the errors are uncorrelated and have different variances, i.e. $\mathrm{E}(\mathbf{e}\mathbf{e}') = \sigma^2\mathbf{G}$, and moreover $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$, where $\mathbf{0}_n$ is an $n \times 1$ null vector, $\mathbf{G}$ is the known $n \times n$ diagonal positive definite matrix. Accordingly, any spring balance weighing design is nonsingular if and only if $\mathbf{X}$ is of full column rank, i.e. $\mathrm{r}(\mathbf{X}) = p$. In such a design for the estimation of unknown weights of p objects, we use the general weighted least squares method and we get

$$\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}. \tag{2.2}$$

The covariance matrix of $\hat{\mathbf{w}}$ is equal to

$$\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}. \tag{2.3}$$

The matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix.

In the literature, some study on optimality criterions are presented. For a deeper discussion we refer to the paper Jacroux and Notz (1983). In many papers concerning the weighing designs, the A-optimal design is considered. For the given covariance matrix of errors $\sigma^2\mathbf{G}$, the design $\mathbf{X}$ is A-optimal if the sum of variances of estimators for unknown parameters is minimal, i.e. $\mathrm{tr}\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ is minimal in the class $\mathbf{\Psi}_{n\times p}\{0,1\}$. Moreover, the design for which the sum of variances of estimators of parameters attains the lowest bound in $\mathbf{\Psi}_{n\times p}\{0,1\}$ is called the regular A-optimal design. Let us note, in the set of design matrices $\mathbf{\Psi}_{n\times p}\{0,1\}$, the regular A-optimal design may not exist, whereas A-optimal design always exist. The concept of the A-optimality was shown in Pukelsheim (1983), Shah and Sinha (1989), Ceranka and Graczyk (2004), Ceranka, Graczyk, Katulska (2006, 2007), Masaro and Wong (2008), Graczyk (2011, 2012).

In this paper we consider the experimental situation where we determine unknown measurements of p objects in $n = \sum_{s=1}^{h} n_s$ measurement operations under model (2.1). It is assumed that n_s measurements are taken in different h conditions or at different h installations, $s = 1,2,...,h$. So, the covariance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix $\mathbf{G}$

$$\mathbf{G} = \begin{bmatrix} g_1^{-1}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} & \cdots & \mathbf{0}_{n_1}\mathbf{0}'_{n_h} \\ \mathbf{0}_{n_2}\mathbf{0}'_{n_1} & g_2^{-1}\mathbf{I}_{n_2} & \cdots & \mathbf{0}_{n_2}\mathbf{0}'_{n_h} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{n_h}\mathbf{0}'_{n_1} & \mathbf{0}_{n_h}\mathbf{0}'_{n_2} & \cdots & g_h^{-1}\mathbf{I}_{n_h} \end{bmatrix}, \tag{2.4}$$

where $g_s > 0$ denotes the factor of precision, $s = 1,2,...,h$. Consequently, according to the partition of $\mathbf{G}$ we write the design matrix $\mathbf{X} \in \mathbf{\Psi}_{n\times p}\{0,1\}$ as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \cdots \\ \mathbf{X}_h \end{bmatrix}, \tag{2.5}$$

where $\mathbf{X}_s$ is the $n_s \times p$ design matrix of any spring balance weighing design. Graczyk (2012) gave the following theorems and definition.

Theorem 2.1. Let p be odd. In any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Psi}_{n\times p}\{0,1\}$ in (2.5) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (2.4),

$$\mathrm{tr}\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1} \geq \frac{4p^3}{(p+1)^2\,\mathrm{tr}\left(\mathbf{G}^{-1}\right)}. \tag{2.6}$$

Definition 2.1. Let p be odd. Any $\mathbf{X} \in \mathbf{\Psi}_{n\times p}\{0,1\}$ in (2.5) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2.4), is said to be the regular A-optimal spring balance weighing design if

$$\mathrm{tr}\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1} = \frac{4p^3}{(p+1)^2\,\mathrm{tr}\left(\mathbf{G}^{-1}\right)}. \tag{2.7}$$

Theorem 2.2. Let p be odd. Any $\mathbf{X} \in \mathbf{\Psi}_{n\times p}\{0,1\}$ in (2.5) with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2.4), is the regular A-optimal spring balance weighing design if and only if

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{p+1}{4p}\,\mathrm{tr}\left(\mathbf{G}^{-1}\right)\left(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p'\right). \tag{2.8}$$

3. Construction of the regular A-optimal designs

In Graczyk (2012) some construction methods of the regular A-optimal spring balance weighing design are given. In this paper we present some new experimental plans of such designs. The construction of a regular A-optimal spring balance weighing design proceeds as follow. It is worth pointing out the incidence matrices of the block designs may be used for the construction of the design matrix $\mathbf{X} \in \mathbf{\Psi}_{n \times p}\{0,1\}$, then we take $n = \sum_{s=1}^{h} b_s$ and $p = v$. Now, we present some series of the balanced incomplete block designs. Based on their incidence matrices we form the design matrix $\mathbf{X} \in \mathbf{\Psi}_{n \times p}\{0,1\}$ of the spring balance weighing design with $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (2.4), that is the regular A-optimal design. Summarizing, we can formulate our main result.

Theorem 3.1. Let v be odd and let $\mathbf{N}$ be the incidence matrix of balanced incomplete block design with the parameters

(i) $v = b = 4t+3$, $r = k = 2(t+1)$, $\lambda = t+1$, where $4t+3$ is a prime or a prime power (the complementary design to the design which is described in Raghavarao (1971, Theorem 5.7.4) or Raghavarao and Padgett (2005, Corollary 4.15.2),

(ii) $v = 4t+1$, $b = 2(4t+1)$, $r = 2(2t+1)$, $k = 2t+1$, $\lambda = 2t+1$, where $4t+1$ is a prime or a prime power (the complementary design to the design which is described in Raghavarao (1971, Theorem 5.75),

(iii) $v = b = 4(k^*)^2 - 1$, $r = k = 2(k^*)^2$, $\lambda = (k^*)^2$, (if there exists a balanced incomplete block design with $\lambda^* = 1$ and $r^* = 2k^* + 1$, Raghavarao (1971, Theorem 5.9.2) or Raghavarao and Padgett (2005, Theorem 4.13),

(iv) v, $b = \binom{v}{t}$, $r = \binom{v-1}{t-1}$, $k = t$, $\lambda = \binom{v-2}{t-2}$ of an irreducible balance incomplete block design, Raghavarao (1971, p. 90), or Raghavarao and Padgett (2005, p. 86).

Any $\mathbf{X} \in \mathbf{\Psi}_{hb \times v}\{0,1\}$ in the form $\mathbf{X} = \mathbf{1}_h \otimes \mathbf{N}'$ is the regular A-optimal spring balance weighing design with the covariance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (2.4).

Proof. For the design matrix $\mathbf{X} = \mathbf{1}_h \otimes \mathbf{N}'$ and $\mathbf{G}$ in (2.4), we have $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mathbf{NN}'\sum_{s=1}^{h} g_s$. $\mathbf{N}$ is the incidence matrix of balanced incomplete block design therefore $\mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$. We have $\mathbf{NN}' = \lambda(\mathbf{I}_v + \mathbf{1}_v\mathbf{1}_v')$ because $r = 2\lambda$. On the account of the above remark, we get $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \lambda\sum_{s=1}^{h} g_s(\mathbf{I}_v + \mathbf{1}_v\mathbf{1}_v')$. So, it is evident that the condition (2.8) holds.

4. Example

As an example let us consider the experiment in that we determine unknown measurements of $p = 5$ objects using $n = 20$ measuring operations. The covariance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix $\mathbf{G} = \begin{bmatrix} g_1^{-1}\mathbf{I}_{10} & \mathbf{0}_{10}\mathbf{0}_{10}' \\ \mathbf{0}_{10}\mathbf{0}_{10}' & g_2^{-1}\mathbf{I}_{10} \end{bmatrix}$, where $g_1, g_2 > 0$. To construct the design matrix $\mathbf{X} \in \mathbf{\Psi}_{20\times 5}\{0,1\}$ we can use the balanced incomplete block design with the parameters $v = 5$, $b = 10$, $r = 6$, $k = 3$, $\lambda = 3$ given by the incidence matrix

$$\mathbf{N} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

In this case we have $\mathbf{X} = \begin{bmatrix} \mathbf{N}' \\ \mathbf{N}' \end{bmatrix}$ and $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = 3(g_1 + g_2)(\mathbf{I}_5 + \mathbf{1}_5\mathbf{1}_5')$. From (2.8) we obtain $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{6\cdot 10}{4\cdot 5}(g_1 + g_2)(\mathbf{I}_5 + \mathbf{1}_5\mathbf{1}_5')$. Hence

$$(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \frac{1}{3(g_1 + g_2)}\left(\mathbf{I}_5 - \frac{1}{6}\mathbf{1}_5\mathbf{1}_5'\right),$$

therefore $\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1} = \frac{25}{18(g_1 + g_2)}$. The same conclusion can be drawn from (2.7). In this case we obtain

$$\mathrm{tr}\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1} = \frac{4 \cdot 5^3}{10 \cdot 6^2 (g_1 + g_2)} = \frac{25}{18(g_1 + g_2)}.$$

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