Colloquium Biometricum 43
2013, 59–72

A METHOD OF CONSTRUCTING INCOMPLETE SPLIT-SPLIT-PLOT DESIGNS SUPPLEMENTED BY WHOLE PLOT AND SUBPLOT STANDARDS AND THEIR ANALYSIS

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In Honour of Professor Tadeusz Caliński on the Occasion of his 85th Birthday

Summary

In the paper we consider a situation when split-split-plot experimental designs are incomplete due to both whole plot treatments and subplot treatments only. With respect to the sub-subplot treatments the presented designs are always orthogonal. Additionally one of the whole plot treatments and one of the subplot treatments are treated as standards.

To construct the final three-factor designs with nested treatment structures we can use two different or the same block designs from the class of orthogonally supplemented PEB block designs with at most $(m+1)$ – efficiency classes. We present also algebraic and statistical properties of the generating designs and the resulting from the construction method split-split-plot designs as well as a numerical example.

Keywords and phrases: general balance, partial efficiency, standard treatment, supplemented block design, split-split-plot design, stratum efficiency factors

Classification AMS 2010: 62K10, 62K15

1. Introduction

In conducting agricultural experiments with three or more different factors complete versions of split-split-plot design (say, SSP design) are often used. Thus, we assume that its methodology of planning and analysis is well known.

In statistical terms, each complete experimental design, also known as orthogonal design, is "the best". Orthogonality of that design facilitates the optimal estimation of treatment and interaction effects, and enables one to easily perform testing of both general and specific hypotheses. Sometimes however, in a practice, a limitation of an experimental material or / and economic considerations do not allow to set up a complete experiment in the SSP design. This is the case for example in testing of varieties, when the number of tested varieties is very large. This often implies that blocks, within which the treatments are randomly distributed, do not behave homogeneity. Failure to comply with the known and fundamental principle of a blocking of the units, can substantially change the analysis and lead to misleading conclusions from the experiment. In such situation we can plan a non-orthogonal experiment and conduct it in an appropriately selected incomplete version of the split-split-plot design, i.e. such, within which not all of the treatment combinations are inside blocks. This subject was also raised in the paper Mejza and Mejza (1997c).

Purpose of the present paper is presenting new method of the constructing non-orthogonal SSP designs which are incomplete due to both whole plot treatments and subplot treatments whereas complete with respect to the sub-subplot treatments only. Additionally one of the whole plot treatments and one of the subplot treatments are treated as standards.

Mejza (1997a, 1997b) considered modeling incomplete SSP designs, their statistical properties and further consequences for the analysis. Other methods of the constructing incomplete SSP designs and their statistical properties can be also found in Ambroży and Mejza (2011, 2012).

In the present construction of the incomplete SSP designs some generating block designs for the whole plot treatments and subplot treatments are used. They come from the class of orthogonally supplemented PEB block designs with at most $(m + 1)$ – efficiency classes (see, Caliński 1971, Caliński and Ceranka 1974, Puri and Nigam 1977, Puri et al. 1977, Nigam and Puri 1982, Kachlicka and Mejza 1998, Caliński and Kageyama 2003 Sections 6.3. and 10.3.3). Additionally, we assume both generating designs can be the same or different. Other sub-subplot treatments are randomly arranged in a randomized complete block (RCB) design.

2. Assumptions and notations

Let us consider a three-factor experiment in which the first factor, say A, has s levels $A_1, A_2, \ldots, A_s$ (called also the whole plot treatments), the second factor, say B, has t levels $B_1, B_2, \ldots, B_t$ (called the subplot treatments) and the third factor, say C, has w levels $C_1, C_2, \ldots, C_w$ (called the sub-subplot treatments). Thus the number $\nu = stw$ denotes the number of all treatment combinations in the experiment.

There is assumed the experimental material can be divided into b blocks with $k_A < s$ whole plots. Then, each whole plot is divided into $k_B < t$ subplots with $k_C = w$ sub-subplots. The s whole plot (A) treatments are randomly allocated to the whole plots within each block, t subplot (B) treatments are randomly allocated to the subplots within each whole plot, and w (C) sub-subplot treatments are randomly allocated to the sub-subplots within each subplot. Let's note the third factor C is in a split-plot relation to the whole plot and subplot treatment combinations in the SSP design. Next in the paper we adopt the following notation: $\mathbf{1}_x$ is the *x*-dimensional vector of ones, $\mathbf{I}_x$ denotes *x*-dimensional unity matrix.

3. Constructing method of the incomplete SSP designs

This method is based on Kronecker product of three subdesigns, in which the levels of three factors (A, B, C) are assigned. Let's assume that w sub-subplot (C) treatments are in an appropriate RCB design. Whereas the *s* whole plot (A) treatments occur in a supplemented block design d_A ($v_A = s$, b_A, k_A, $\mathbf{r}_A$), wherein the parameters v_A, b_A, k_A are numbers of the whole plot treatments, blocks, units inside each block in the subdesign d_A, respectively and $\mathbf{r}_A$ denotes a vector of replicates of the all whole plot treatments and the t subplot (B) treatments occur in a supplemented block design d_B ($\nu_B = t$, b_B, k_B, $\mathbf{r}_B$), wherein the parameters ν_B, b_B, k_B mean numbers of the whole plot treatments, blocks, units inside each block in the subdesign d_B, respectively and $\mathbf{r}_B$ denotes a vector of replicates of the all subplot treatments. Symbols k_A and k_B are given in section 2.

Additionally, we assume both whole plot (A) treatments and the subplot (B) treatments form two groups: $s = s_1 + 1$ and $t = t_1 + 1$, respectively. The s_1 test (basic) A treatments are allocated in a subdesign $\tilde{d}_A$ which is a partially efficiency balanced (PEB) design with at most m_1 efficiency classes (cf. Puri et al. 1977, Kageyama and Puri 1985, Caliński and Kageyama 2000 Definition 4.3.1.) with incidence matrix $\tilde{\mathbf{N}}_A$ supplemented then by one standard (A) treatment and similarly the t_1 test (basic) B treatments are allocated in a subdesign $\tilde{d}_B$ which is a partially efficiency balanced (PEB) design with at most m_2 efficiency classes with incidence matrix $\tilde{\mathbf{N}}_B$ also supplemented then by one standard (B) treatment. Let $\tilde{\mathbf{N}}_A$ be the $s_1 \times \tilde{b}_A$ incidence matrix of the subdesign $\tilde{d}_A$ with parameters:

$$s_1,\ \tilde{b}_A,\ \ \tilde{k}_A,\ \mathbf{r}_{s_1} = [r_1, r_2, ..., r_{s_1}]'\ ,\ \tilde{\varepsilon}_j^A,\ \tilde{\rho}_j^A\ \left(\sum_{j=1}^{m_1} \tilde{\rho}_j^A = s_1 - 1 \right), \tag{3.1}$$

which define number of test A treatments, number of blocks, size of blocks, vector of test A treatment replicates, as well as eigenvalues and their multiplicities of so-called $\mathbf{C}$ matrix of the subdesign $\tilde{d}_A$, respectively.

Likewise, let $\tilde{\mathbf{N}}_B$ be the $t_1 \times \tilde{b}_B$ incidence matrix of the subdesign $\tilde{d}_B$ with parameters:

$$t_1,\ \tilde{b}_B,\ \ \tilde{k}_B,\ \mathbf{r}_{t_1} = [r_1, r_2, ..., r_{t_1}]',\ \tilde{\varepsilon}_l^B,\ \tilde{\rho}_l^B\ \left(\sum_{l=1}^{m_2} \tilde{\rho}_l^B = t_1 - 1 \right), \tag{3.2}$$

which define number of test B treatments, number of blocks, size of blocks, vector of test B treatment replicates, and eigenvalues and their multiplicities of so-called $\mathbf{C}$ matrix of the subdesign $\tilde{d}_B$, respectively. So, respecting both kinds of treatments for the factors A and B, and using (3.1) - (3.2), the incidence matrices can be written as follows:

$$\mathbf{N}_A = \begin{bmatrix} \tilde{\mathbf{N}}_A \\ \mathbf{1}'_{\tilde{b}_A} \end{bmatrix},\ \ \mathbf{N}_B = \begin{bmatrix} \tilde{\mathbf{N}}_B \\ \mathbf{1}'_{\tilde{b}_B} \end{bmatrix}. \tag{3.3}$$

They are incidence matrices of the PEB designs with at most (m_1+1) – and (m_2+1) classes of efficiency with parameters:

– for the whole plot treatments:

$$v_A = s = s_1 + 1, \quad b_A = \tilde{b}_A, \quad \mathbf{k}_A = k_A \mathbf{1}_{b_A} = (\tilde{k}_A + 1)\mathbf{1}_{b_A},$$

$$\mathbf{r}_A = \mathbf{N}_{d_A}\mathbf{1}_s = [\mathbf{r}'_{s_1}, b_A]' = [r_1, r_2, ..., r_{s_1}, b_A]', \tag{3.4}$$

$$\varepsilon_0^A = 1, \ \rho_0^A = 1, \ \varepsilon_j^A = 1 - (\tilde{n}_A / n_A)(1 - \tilde{\varepsilon}_j^A), \ \rho_j^A = \tilde{\rho}_j^A, \ j = 1,2,\ldots,m_1$$

where $\tilde{n}_A$ and n_A denote numbers of observations in the designs $\tilde{d}_A$ and d_A, respectively, while ε_j^A, ρ_j^A, $(j = 1,2,\ldots,m_1)$, mean eigenvalues and their multiplicities of so-called **C** matrix of the subdesign d_A;

– for the subplot treatments:

$$v_B = t = t_1 + 1, \quad b_B = \tilde{b}_B, \quad \mathbf{k}_B = k_B \mathbf{1}_{b_B} = (\tilde{k}_B + 1)\mathbf{1}_{b_B},$$

$$\mathbf{r}_B = \mathbf{N}_{d_B}\mathbf{1}_t = [\mathbf{r}'_{t_1}, b_B]' = [r_1, r_2, ..., r_{t_1}, b_B]', \tag{3.5}$$

$$\varepsilon_0^B = 1, \quad \rho_0^B = 1, \quad \varepsilon_j^B = 1 - (\tilde{n}_B / n_B)(1 - \tilde{\varepsilon}_j^B), \quad j = 1,2,\ldots,m_2$$

where $\tilde{n}_B$ and n_B denote numbers of observations in the designs $\tilde{d}_B$ and d_B, respectively as well as ε_j^B, ρ_j^B, $j = 1,2,\ldots,m_2$, mean eigenvalues and their multiplicities of so-called **C** matrix of the subdesign d_B.

We use foregoing information about the generating subdesigns in the constructing method of an incomplete SSP design.

Let $\mathbf{N}_1 = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{1}_w$ be the $v \times b$ incidence matrix of the considered SSP design with parameters:

$$\nu = stw, \ b = b_A b_B, \ k = k_A k_B w, \ \mathbf{r} = \mathbf{r}_A \otimes \mathbf{r}_B \otimes \mathbf{1}_w \tag{3.6}$$

where matrices $\mathbf{N}_A$ and $\mathbf{N}_B$ are given in (3.3) and their parameters are presented in (3.4)–(3.5). This incidence matrix $\mathbf{N}_1$ with respect to blocks plays an important role in construction methods of any SSP design. The applied construction leads to proper (cf. Caliński and Kageyama 2000, Definition 2.2.2) and non-equireplicated SSP design (cf. Caliński and Kageyama 2000, Definition 2.2.3).

The orthogonal block structure of the considered SSP design allows one to apply Nelder's approach to the analysis of variance for the multistratum experiments (Nelder 1965a, 1965b). The stratum analyses are expressed in terms of basic contrasts introduced by Pearce et al. (1974).

It can be shown (e.g. Ambroży and Mejza, 2011) that in the SSP stratum model there are five strata, i.e. the total-area stratum ("zero" stratum), the inter-block stratum (the first stratum), the inter-whole plot stratum (the second stratum), the inter-subplot stratum (the third stratum) and the inter-sub-subplot stratum (the fourth stratum).

Statistical properties of the SSP designs are related mainly to algebraic properties of stratum information matrices $\mathbf{A}_f$, $f = 0,1,\ldots,4$ (cf. Ambroży and Mejza, 2011). In the present case of the design, forms of these matrices are given in (3.7).

Assuming that $\mathbf{r}_A^\delta = \mathrm{diag}(r_1, r_2, ..., r_{s_1}, b_A)$ and $\mathbf{r}_B^\delta = \mathrm{diag}(r_1, r_2, ..., r_{t_1}, b_B)$ we have

$$\begin{aligned}
\mathbf{A}_0 &= \frac{1}{bk}\mathbf{r}_A\mathbf{r}_A' \otimes \mathbf{r}_B\mathbf{r}_B' \otimes \mathbf{1}_w\mathbf{1}_w', \\
\mathbf{A}_1 &= \frac{1}{k}\mathbf{N}_A\mathbf{N}_A' \otimes \mathbf{N}_B\mathbf{N}_B' \otimes \mathbf{1}_w\mathbf{1}_w' - \frac{1}{bk}\mathbf{r}_A\mathbf{r}_A' \otimes \mathbf{r}_B\mathbf{r}_B' \otimes \mathbf{1}_w\mathbf{1}_w', \\
\mathbf{A}_2 &= \frac{1}{k_2 w}\left(\mathbf{r}_A^\delta - \frac{1}{k_1}\mathbf{N}_A\mathbf{N}_A'\right) \otimes \mathbf{N}_B\mathbf{N}_B' \otimes \mathbf{1}_w\mathbf{1}_w', \\
\mathbf{A}_3 &= \frac{1}{w}\mathbf{r}_A^\delta \otimes \left(\mathbf{r}_B^\delta - \frac{1}{k_2}\mathbf{N}_B\mathbf{N}_B'\right) \otimes \mathbf{1}_w\mathbf{1}_w', \\
\mathbf{A}_4 &= \mathbf{r}_A^\delta \otimes \mathbf{r}_B^\delta \otimes \left(\mathbf{I}_w - \frac{1}{w}\mathbf{1}_w\mathbf{1}_w'\right).
\end{aligned} \tag{3.7}$$

One can check that the resulting SSP design is generally balanced. It follows from the fact the matrices (3.7) commute with respect to $\mathbf{r}^{-\delta}$ (e.g. Mejza 1992, Ambroży and Mejza 2011), where $\mathbf{r}^{-\delta} = \mathrm{diag}(1/r_1, ..., 1/r_v)$. This means that the matrices (3.7) have a common set of eigenvectors $\mathbf{p}_h$ corresponding to eigenvalues ε_{fh} with respect to $\mathbf{r}^{\delta}$, where $0 \le \varepsilon_{fh} \le 1$ and $f = 0, 1, 2, 3, 4$; $h = 1, 2,..., v$. Since $\mathbf{A}_f \mathbf{1}_v = \mathbf{0}$ for $f > 0$, the last eigenvector may be chosen as $\mathbf{p}_v = n^{-1/2}\mathbf{1}_v$. The remaining eigenvectors $\mathbf{p}_h$ for $h < v$, where $\mathbf{p}'_h \mathbf{r}^{\delta} \mathbf{p}_h = 0$, form the basis for all vectors generating some contrasts. We can note that any vector $\mathbf{c}_h = \mathbf{r}^{\delta}\mathbf{p}_h$ such that the eigenvector $\mathbf{p}_h$ satisfies the condition

$$\mathbf{A}_f \mathbf{p}_h = \varepsilon_{fh} \mathbf{r}^{\delta} \mathbf{p}_h , \quad \text{for} \quad f = 0, 1, 2, 3, 4; \; h = 1, 2,..., v \tag{3.8}$$

defines an orthogonal (basic) contrast $\mathbf{c}'_h \boldsymbol{\tau}$ (cf. Pearce et al., 1974).

It allows to define a common set of basic contrasts $\mathbf{c}'_h \boldsymbol{\tau}$ and corresponding to them stratum efficiency factors ε_{fh} which satisfy the following relations (e.g. Mejza, 1997a):

$$\underset{h<v}{\forall} (\varepsilon_{0v} = 1, \; \varepsilon_{0h} = 0), \qquad \sum_{f=1}^{4} \varepsilon_{fh} = 1, \; \text{ for } \; h < v .$$

If $\varepsilon_{fh} = 1$ then full information on the *h*-th basic contrast is included in one stratum (the *f*-th stratum) only. We can say the SSP design is orthogonal in the *f*-th stratum with respect to this contrast. If $0 < \varepsilon_{fh} < 1$ then the information on the *h*-th basic contrast occurs in two strata at least. From the relation $\mathbf{A}_0 \mathbf{1}_v \neq \mathbf{0}$ it follows that the total-area stratum (for $f = 0$) is connected mainly with estimating the general mean.

Table 1. Stratum efficiency factors of the considered incomplete SSP design

Types of contrasts	**df**	**Strata**			
		1	**2**	**3**	**4**
A^T	$\left.\begin{matrix}\rho_1^A \\ \cdots \\ \rho_{m_1}^A\end{matrix}\right\} = s_1 - 1$	$\begin{matrix}1-\varepsilon_1^A \\ \cdots \\ 1-\varepsilon_{m_1}^A\end{matrix}$	$\begin{matrix}\varepsilon_1^A \\ \cdots \\ \varepsilon_{m_1}^A\end{matrix}$		
A^T *vs.* A^{SD}	$\rho_0^A = 1$		$\varepsilon_0^A = 1$		
B^T	$\left.\begin{matrix}\rho_1^B \\ \cdots \\ \rho_{m_2}^B\end{matrix}\right\} = t_1 - 1$	$\begin{matrix}1-\varepsilon_1^B \\ \cdots \\ 1-\varepsilon_{m_2}^B\end{matrix}$		$\begin{matrix}\varepsilon_1^B \\ \cdots \\ \varepsilon_{m_2}^B\end{matrix}$	
B^T *vs.* B^{SD}	$\rho_0^B = 1$			$\varepsilon_0^B = 1$	
$A^T \times B^T$	$\left.\begin{matrix}\rho_1^A\rho_1^B \\ \vdots \\ \rho_1^A\rho_{m_2}^B \\ \cdots \\ \rho_{m_1}^A\rho_1^B \\ \vdots \\ \rho_{m_1}^A\rho_{m_2}^B\end{matrix}\right\} =$ $=(s_1-1)(t_1-1)$	$\begin{matrix}(1-\varepsilon_1^A)(1-\varepsilon_1^B) \\ \vdots \\ (1-\varepsilon_1^A)(1-\varepsilon_{m_2}^B) \\ \cdots\cdots \\ (1-\varepsilon_{m_1}^A)(1-\varepsilon_1^B) \\ \vdots \\ (1-\varepsilon_{m_1}^A)(1-\varepsilon_{m_2}^B)\end{matrix}$	$\begin{matrix}\varepsilon_1^A(1-\varepsilon_1^B) \\ \vdots \\ \varepsilon_1^A(1-\varepsilon_{m_2}^B) \\ \cdots\cdots\cdots \\ \varepsilon_{m_1}^A(1-\varepsilon_1^B) \\ \vdots \\ \varepsilon_{m_1}^A(1-\varepsilon_{m_2}^B)\end{matrix}$	$\begin{matrix}\varepsilon_1^B \\ \vdots \\ \varepsilon_{m_2}^B \\ \cdots\cdots \\ \varepsilon_1^B \\ \vdots \\ \varepsilon_{m_2}^B\end{matrix}$	
$(A^T$ *vs.* $A^{SD})$ $\times B^T$	$\left.\begin{matrix}\rho_0^A\rho_1^B \\ \cdots \\ \rho_0^A\rho_{m_2}^B\end{matrix}\right\} = t_1 - 1$		$\begin{matrix}\varepsilon_0^A(1-\varepsilon_1^B) \\ \cdots \\ \varepsilon_0^A(1-\varepsilon_{m_2}^B)\end{matrix}$	$\begin{matrix}\varepsilon_1^B \\ \cdots \\ \varepsilon_{m_2}^B\end{matrix}$	
$A^T \times$ $(B^T$ *vs.* $B^{SD})$	$\left.\begin{matrix}\rho_1^A\rho_0^B \\ \cdots \\ \rho_{m_1}^A\rho_0^B\end{matrix}\right\} = s_1 - 1$			$\varepsilon_0^B = 1$	
$(A^T$ *vs.* $A^{SD})$ $\times$ $(B^T$ *vs.* $B^{SD})$	$\rho_0^A\,\rho_0^B = 1$			$\varepsilon_0^B = 1$	
C	$w-1$				1

Table 1. continued

$A^T \times C$	$\left.\begin{matrix}\rho_1^*(w-1)\\ \cdots \\ \rho_{m_1}^*(w-1)\end{matrix}\right\}= =(s_1-1)(w-1)$				1
$(A^T\ vs.\ A^{SD}) \times C$	$\rho_0^A(w-1) = = w-1$				$\varepsilon_0^A = 1$
$B^T \times C$	$\left.\begin{matrix}\rho_1^*(w-1)\\ \cdots \\ \rho_{m_2}^*(w-1)\end{matrix}\right\}= =(t_1-1)(w-1)$				1
$(B^T\ vs.\ B^{SD}) \times C$	$\rho_0^B(w-1) = = w-1$				$\varepsilon_0^B = 1$
$A^T \times B^T \times C$	$\left.\begin{matrix}\rho_1^A\rho_1^B(w-1)\\ \vdots \\ \rho_1^A\rho_{m_2}^B(w-1)\\ \cdots\cdots\cdots \\ \rho_{m_1}^B\rho_1^B(w-1)\\ \vdots \\ \rho_{m_1}^B\rho_{m_2}^B(w-1)\end{matrix}\right\}= =(s_1-1)(t_1-1)(w-1)$				1
$A^T \times (B^T\ vs.\ B^{SD}) \times C$	$(s_1-1)\rho_0^B(w-1) = = (s_1-1)(w-1)$				$\varepsilon_0^B = 1$
$(A^T\ vs.\ A^{SD}) \times B^T \times C$	$\rho_0^A(t_1-1)(w-1) = =(t_1-1)(w-1)$				$\varepsilon_0^A = 1$
$(A^T\ vs.\ A^{SD}) \times (B^T\ vs.\ B^{SD}) \times C$	$\rho_0^A\rho_0^B(w-1) = = w-1$				$\varepsilon_0^A\varepsilon_0^B = 1$

df (*degrees of freedom*) – numbers of the particular types of the contrasts estimable in the strata; 1 – the inter-block stratum, 2 – the inter-whole plot stratum, 3 – the inter-subplot stratum, 4 – the inter-sub-subplot stratum

In the present paper, we consider the following types of the basic contrasts: among main effects of the whole plot treatments including: the test *A* treatments (A^T) and between the test group and the whole plot standard (A^T *vs.* A^{SD}), then among main effects of the subplot (*B*) treatments including: the test *B* treatments (B^T) and between the test group and the whole plot standard (B^T *vs.* B^{SD}), among main effects of the sub-subplot (*C*) treatments, and other interaction contrasts as in Table 1.

Analyzing algebraic properties of the matrices (3.7) we can obtain information about estimability of the basic contrasts in the strata and their stratum efficiency factors ε_{fh}, $f = 1,..., 4$; $h = 1, 2,..., v-1 = stw-1$. In Table 1 the ε_{fh} are expressed by the eigenvalues ε_j^A, $j = 1,2,\ldots,m_1$ and ε_l^B, $l = 1,2,\ldots,m_2$ given in (3.4) and (3.5) according to the construction method.

4. Example

Let us assume the aim of the experiment was to investigate the reaction of *s* = 7 genotypes of winter wheat (the whole plot treatments) for *t* =5 different doses of nitrogen fertilization (the subplot treatments) and a chemical preparation – growth regulator (*w* = 2). So we have $v = stw = 70$ treatment combinations.

The genotypes comprised six new varieties ($s_1 = 6$; $A_1, ..., A_6$) called the test whole plot (*A*) treatments and one standard variety (A_7) called the standard whole plot (*A*) treatment. The test subplot (*B*) treatments were definited by increasing fertilization doses: B_1, B_2, B_3, B_4 ($t_1 = 4$), and B_5 (no fertilization) signified the standard (control) subplot treatment. The sub-subplot treatments corresponded to the application (or no application) of the chemical preparation.

Because an experimental material connected with new varieties was limited, this experiment was conducted in an incomplete split-split-plot (SSP) design with an incidence matrix $\mathbf{N}_1 = \mathbf{N}_A \otimes \mathbf{N}_B \otimes \mathbf{1}_w$. Before specifying the parameters of the SSP design, the statistical properties of the generating subdesigns for factors *A* and *B* will be discussed.

It was assumed that

$$\mathbf{N}_A = \begin{bmatrix} \tilde{\mathbf{N}}_A \\ \mathbf{1}_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } \mathbf{N}_B = \begin{bmatrix} \tilde{\mathbf{N}}_B \\ \mathbf{1}_4' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad (4.1)$$

where $\tilde{\mathbf{N}}_{\mathrm{A}}$ and $\tilde{\mathbf{N}}_{\mathrm{B}}$ are incidence matrices of group divisible partially balanced incomplete block designs of types S1 and SR1, respectively (see Clatworthy 1973, Ambroży and Mejza 2005). They apply to the test whole plot treatments and the test subplot treatments. You can write now the statistical properties (3.1) as follows:

$$s_1 = 6, \quad \tilde{b}_A = 3, \quad \tilde{k}_A = 4, \quad \mathbf{r}_{s_1} = [2, 2, 2, 2, 2, 2]'$$

which define number of the test varieties, number of blocks, size of blocks, vector of test A treatment replicates, as well as eigenvalues $\tilde{\varepsilon}_1^A = 1$ and $\tilde{\varepsilon}_2^A = 0.75$ with their multiplicities $\tilde{\rho}_1^A = 3$ and $\tilde{\rho}_2^A = 2$, respectively, of so-called $\mathbf{C}$ matrix of the subdesign $\tilde{d}_A$.

Likewise, the statistical properties (3.2) are as follows:

$$t_1 = 4, \; \tilde{b}_B = 4, \; \tilde{k}_B = 2, \; \mathbf{r}_{t_1} = [2, 2, 2, 2]',$$

$$\tilde{\varepsilon}_1^B = 1, \; \tilde{\rho}_1^B = 1, \; \tilde{\varepsilon}_2^B = 0.5, \; \tilde{\rho}_2^B = 2.$$

Each of the $\tilde{\mathbf{N}}_{\mathrm{A}}$ and $\tilde{\mathbf{N}}_{\mathrm{B}}$ is supplemented respectively, by one standard treatment, hence we obtain the matrices given in (4.1). They concern the generating designs d_A and d_B which are PEB designs with $m_1 = m_2 = 2$ efficiency classes. The parameters are:

– for the whole plot treatments (all varieties): $s = 7$, $b_A = 3$, $\mathbf{k}_A = 5 \cdot \mathbf{1}_3$, $\mathbf{r}_A = [2, 2, 2, 2, 2, 2, 3]'$, $\varepsilon_0^A = 1$, $\rho_0^A = 1$, $\varepsilon_1^A = 1$, $\rho_1^A = 3$, $\varepsilon_2^A = 0.8$, $\rho_2^A = 2$;

– for the subplot treatments (all doses of fertilization): $t = 5$, $b_B = 4$, $\mathbf{k}_B = 3 \cdot \mathbf{1}_4$, $\mathbf{r}_B = [2, 2, 2, 2, 4]'$, $\varepsilon_0^B = 1$, $\rho_0^B = 1$, $\varepsilon_1^B = 1$, $\rho_1^B = 1$, $\varepsilon_2^B = 0.667$, $\rho_2^B = 2$.

Finally, using (3.6) and the algebraic properties given above, we can write the following parameters of the incomplete SSP design:

$$v = 70, \; b = 12, \; k = 30, \; \mathbf{r} = [2, 2, 2, 2, 2, 2, 3]' \otimes [2, 2, 2, 2, 4]' \otimes \mathbf{1}_2 .$$

We can obtain information on the algebraic properties of the matrices $\mathbf{A}_f$, f = 1,2,3,4 defined in (3.7), resulting from the eigenvalues of the generating matrices (see Table 1) or an appropriate computer program too.

It can be shown that the eigenvalues ε_{fh} from (3.8), where; $f = 1, 2, 3, 4$; $h = 1, 2, \ldots, 69$ (and their multiplicities) for each matrix are as follows:

$\mathbf{A}_1$: 0.333 (2); 0.2 (6); 0.0667 (4);0 (58)

$\mathbf{A}_2$: 1 (4); 0.8 (2); 0.333 (8); 0.266 (4); 0 (52)

$\mathbf{A}_3$: 1 (14); 0.667 (14); 0 (42)

$\mathbf{A}_4$: 1 (35); 0 (35).

The eigenvalues ε_{fh} which fulfill $0 < \varepsilon_{fh} \le 1$ are interpreted as stratum efficiency factors (see Table 2). The other information is also given in Table 2, for example the distribution of information relating to the basic contrasts.

From Table 2 it can be seen that only contrasts among main effects of the test *A* treatments (A^T), the test *B* treatments (B^T) and the contrasts of the interaction effects of type $A^T \times B^T$, (A^T *vs.* A^{SD})$\times B^T$ are estimated with a different precision (two classes of efficiency). The contrasts with the first group of efficiency are estimated with full efficiency (=1) in appropriate strata and the contrasts with the second group with not full efficiency are estimated in two or three strata. The remaining contrasts are estimated as in a complete (orthogonal) SSP design with full efficiency (= 1). It follows from:

- the construction method (the whole plot (A) treatments and the subplot (B) treatments are allocated in the supplemented PEB subdesigns, the sub-subplot (C) treatments in the RCB subdesign),
- the nature of the SSP design (a nested system of units).

Details connected with the statistical stratum analysis with respect to the basic contrasts can be found in, e.g. Ambroży and Mejza (2006).

In the statistical inference about those contrasts, which are estimable within two or three strata, we can use information about them separately from one stratum only or perform the combined estimation and testing for them based on information from the relevant strata (see, Caliński and Kageyama 2000, Sections 3.7–3.8, 5.5).

Table 2. Stratum efficiency factors of the SSP design in the example

Types of contrasts	**df**	**Strata**			
		1	**2**	**3**	**4**
A^T	3 2	 0.2	1 0.8		
A^T *vs.* A^{SD}	1		1		
B^T	1 2	 0.333		1 0.667	
B^T *vs.* B^{SD}	1			1	
$A^T \times B^T$	5 6 4	 0.066	 0.333 0.267	1 0.667 0.667	
$(A^T \textit{ vs. } A^{SD}) \times B^T$	1 2		 0.333	1 0.667	
$A^T \times (B^T \textit{ vs. } B^{SD})$	5			1	
$(A^T \textit{ vs. } A^{SD}) \times (B^T \textit{ vs. } B^{SD})$	1			1	
C	1				1
$A^T \times C$	5				1
$(A^T \textit{ vs. } A^{SD}) \times C$	1				1
$B^T \times C$	3				1
$(B^T \textit{ vs. } B^{SD}) \times C$	1				1
$A^T \times B^T \times C$	15				1
$A^T \times (B^T \textit{ vs. } B^{SD}) \times C$	5				1
$(A^T \textit{ vs. } A^{SD}) \times B^T \times C$	3				1
$(A^T \textit{ vs. } A^{SD}) \times (B^T \textit{ vs. } B^{SD}) \times C$	1				1

Acknowledgement

The paper was partially supported by project No NN310111640 (NCN–Poland).

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