Colloquium Biometricum 47
2017, 1–15

REGULAR D-OPTIMAL WEIGHING DESIGNS WITH NEGATIVE CORRELATIONS OF ERRORS: NEW CLASSES

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Summary

In this paper, the properties of the regular D-optimal chemical balance weighing design are considered. We consider this design under assumption that the measurements errors are equally negative correlated they have the same variances. Here we study the issues regard to the existence conditions of regular D-optimal design. We present the relations between the parameters of such design and construction methods.

Keywords and phrases: balanced bipartite weighing design, chemical balance weighing design, D-optimality, ternary balanced block design

Classification AMS 2010 : 62K05, 62K10

1. Introduction

Here, we study a linear model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e},$$

where $\mathbf{y}$ is an $n\times 1$ vector of observed weights, $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$, the class of $n\times p$ $(n\geq p)$ matrices $\mathbf{X}=(x_{ij})$, $i=1,2,...,n$, $j=1,2,...,p$, of known elements equal to -1, 1 or 0, $\mathbf{w}$ is a $p\times 1$ vector of unknown measurements of objects and $\mathbf{e}$ is an $n\times 1$ vector of random errors. In such model n is the number of measurements and p is number of objects. We assume that $\mathrm{E}(\mathbf{e})=\mathbf{0}_n$ and $\mathrm{Cov}(\mathbf{e})=\sigma^2\mathbf{G}$, where $\mathbf{0}_n$ is vector of zeros, $\sigma>0$ is known parameter, $\mathbf{G}$ is the $n\times n$ symmetric positive definite matrix of known elements given in the form

$$\mathbf{G}=g\left[(1-\rho)\mathbf{I}_n+\rho\mathbf{1}_n\mathbf{1}_n^{'}\right],\quad g>0,\quad \frac{-1}{n-1}<\rho<0, \tag{1.1}$$

where g,ρ are known, $\mathbf{I}_n$ denotes identity matrix of rank n and $\mathbf{1}_n$ denotes $n\times 1$ vector of ones.

The inverse of matrix $\mathbf{G}$ is given as

$$\mathbf{G}^{-1}=\left(g(1-\rho)\right)^{-1}\left[\mathbf{I}_n-\frac{\rho}{1+\rho(n-1)}\mathbf{1}_n\mathbf{1}_n^{'}\right].$$

For the estimation of unknown measurements of objects $\mathbf{w}$ we use the normal equations $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}}=\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$. We said, that the design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ is nonsingular if and only if the matrix $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, i.e. if and only if $\mathbf{X}$ is of full column rank. Assuming that $\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, the generalized least squares estimator of $\mathbf{w}$ is given by $\hat{\mathbf{w}}=\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{y}$ and $\mathrm{Var}(\hat{\mathbf{w}})=\sigma^2\left(\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$. The matrix $\mathbf{M}=\mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix of the design $\mathbf{X}$. Since the purpose of weighing designs is to estimate each of the individual unknown measurements (weights) of objects with whether accuracy it is reasonable to expect the design to give minimal general variance for all estimated weights. So, D-optimality criterion is considered. The optimality problem is concerned with efficient estimation in the sense of D-optimality by a proper choice of the design matrix $\mathbf{X}$ among many at our disposal. As to the notation of D-optimality, the idea is to minimize the determinant of $\mathbf{M}^{-1}$ for each form of $\mathbf{G}$. Different forms of the matrix $\mathbf{G}$ have been considered in the literature. For details, we recommend Jacroux et al. (1983), Masaro and Wong (2008), Katulska and Smaga (2013). Some applications of such designs are given in Banerjee (1975). The

relations between chemical balance weighing designs and the factorial designs are given by Cheng et al. (2004). The practical applications of such designs in the agriculture, medicine as well as in industy are presented in Bose and Bagchi (2007), Jacroux (2009).

In present paper, we present new results related to the D-optimal chemical balance weighing designs assuming that the random errors are equally negative correlated and with the same variances. We give new construction method of D-optimal design. It is based on the incidence matrices of the balanced bipartite weighing designs and the ternary balanced block designs. We give the lower bound for the determinant of the inverse of the information matrix and the list of the parameters of D-optimal experimental plans.

2. D-optimal design

Let us consider $\mathbf{X}=\left[\mathbf{x}_1,\mathbf{x}_2,...,\mathbf{x}_p\right]\in\mathbf{\Phi}_{n\times p}(-1,0,1)$. Our goal is to determine matrix $\mathbf{X}$, for that the lower bound of $\det\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ is minimal among matrices in the class $\mathbf{\Phi}_{n\times p}(-1,0,1)$. Based on the results given in Rao 1973: Section 1c.1 (ii) (b) we get

Lemma 2.1. For diagonal elements of the inverse of information matrix, the inequality $M_{jj}^{-1}=\left(\mathbf{x}_j'\mathbf{G}^{-1}\mathbf{x}_j\right)^{-1}\geq g(1-\rho)\left(\mathbf{x}_j'\mathbf{x}_j-\frac{\mathbf{x}_j'\mathbf{1}_n\mathbf{1}_n'\mathbf{x}_j\rho}{1+\rho(n-1)}\right)^{-1}$ holds.

Next, we prove the inequality which gives the lower bound for determinant $\mathbf{M}^{-1}$.

From Ceranka and Graczyk (2016) we have

Theorem 2.1. If $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(-1,0,1)$ and $\mathbf{G}$ is given in 1.1 then

$$\det\mathbf{M}^{-1}\geq\left(g(1-\rho)\left(m-\frac{\rho(m-2u)^2}{1+\rho(n-1)}\right)^{-1}\right)^p, \tag{2.1}$$

where $m = \max\{m_1, m_2, ..., m_p\}$, m_j represents the number of elements equal to -1 and 1 in j^{th} column of $\mathbf{X}$, $u = \min\{u_1, u_2, ..., u_p\}$, u_j represents the number of elements equal to -1 in j^{th} column of $\mathbf{X}$, $j = 1,2,..., p$.

Definition 2.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in 1.1, is said to be regular D-optimal if it satisfies the equality in 2.1, that is

$$\det \mathbf{M}^{-1} = \left(g(1-\rho)\left(m - \frac{\rho(m-2u)^2}{1+\rho(n-1)} \right)^{-1} \right)^p .$$

Theorem 2.2. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in 1.1 is regular D-optimal if and only if

(i) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p - \frac{\rho(m-2u)^2}{1+\rho(n-1)}\left(\mathbf{I}_p - \mathbf{1}_p\mathbf{1}_p'\right)$ and

(ii) $\mathbf{X}'\mathbf{1}_n = \mathbf{z}_p$,

where $\mathbf{z}_p$ is $p\times 1$ vector, for which the j^{th} element is equal to $m-2u$ or $-(m-2u)$, $j = 1,2,..., p$.

3. Construction of regular D-optimal designs

In any class $\mathbf{\Phi}_{n\times p,m}\{-1, 0, 1\}$ we are not able to determine regular D-optimal design, for example based on the methods given in Ceranka and Graczyk (2014 a, b, c, 2015). So, the basic idea of this paper is to determine regular D-optimal design in the classes in that it is impossible yet. The construction presented here is based on the incidence matrices of the balanced bipartite weighing designs and the ternary balanced block designs. Here, we broaden the list of classes $\mathbf{\Phi}_{n\times p}(-1,0,1)$ in that regular D-optimal chemical balance weighing design exists. Thus, we recall the definition of the balanced bipartite weighing design

given in Huang (1976) and of the ternary balanced block design given by Billington (1984).

The balanced bipartite weighing design with the parameters $v, b, r, k_1, k_2, \lambda_1, \lambda_2$ there is the design which describes how to replace v treatments in b blocks such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, where $k = k_1 + k_2$. Each treatment appears in r blocks. Every pair of treatments from different subblocks appears together in λ_1 blocks and every pair of treatments from the same subblock appears together in λ_2 blocks. The parameters are related by the following identities $vr = b(k_1 + k_2)$, $b = 0.5\lambda_1 v(v-1)(k_1 k_2)^{-1}$, $r = 0.5\lambda_1(k_1 + k_2)(v-1)(k_1 k_2)^{-1}$, $\lambda_2 = 0.5(\lambda_1(k_1(k_1-1) + k_2(k_2-1)))(k_1 k_2)^{-1}$. If $k_1 \neq k_2$, then each object occurs in r_1 blocks in the first subblock and in r_2 blocks in the second subblock, $r_1 + r_2 = r$ and $r_1 = 0.5\lambda_1(v-1)k_2^{-1}$, $r_2 = 0.5\lambda_1(v-1)k_1^{-1}$. $\mathbf{N}^*$ is the incidence matrix of such a design with the elements equal to 0 or 1 and $\mathbf{N}^*(\mathbf{N}^*)' = (r - \lambda_1 - \lambda_2)\mathbf{I}_v + (\lambda_1 + \lambda_2)\mathbf{1}_v\mathbf{1}_v'$.

Any ternary balanced block design with the parameters v, b, r, k, λ, ρ_1, ρ_2 is a design that describe how to replace v treatments in b blocks, each of size k in such a way that each treatment appears 0, 1 or 2 times in r blocks. Each of the distinct pairs of treatments appears λ times. Each treatment occurs alone in ρ_1 blocks and is repeated two times in ρ_2 blocks, where ρ_1 and ρ_2 are constant for the design. It is straightforward to verify that $vr = bk$, $r = \rho_1 + 2\rho_2$, $\lambda(v-1) = \rho_1(k-1) + 2\rho_2(k-2)$. $\mathbf{N}$ is the incidence matrix of such design with elements equal to 0, 1 or 2 and $\mathbf{N}\mathbf{N}' = (\rho_1 + 4\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$.

Now, we form the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ of the chemical balance weighing design as

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' \\ \mathbf{N}_2' - \mathbf{1}_{b_2}\mathbf{1}_v' \end{bmatrix}, \tag{3.1}$$

where $\mathbf{N}_1$ is constructed from $\mathbf{N}_1^*$ - the incidence matrix of the balanced bipartite weighing design with the parameters $v, b_1, r_1, k_{11}, k_{21}, \lambda_{11}, \lambda_{21}$ by replacing in each column k_{11} elements equal +1 that correspond to the elements belonging to the first subblock by –1 and $\mathbf{N}_2$ is the incidence matrix of the ternary balanced block design with the parameters v, b_2, r_2, k_2, λ_2, ρ_{12}, ρ_{22}. For $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form 3.1 $n = b_1 + b_2$ and $p = v$.

Lemma 3.1. Any chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ in the form 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in 1.1, is nonsingular if and only if $k_{11} \neq k_{21}$ or $k_2 \neq v$.

The optimality conditions indicated in Theorem 2.2 implies that the construction methods of the regular D-optimal design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ are depended on the parameter ρ. Thus, we can formulate the following Theorem.

Theorem 3.1. Any nonsingular chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form 1.1, is regular D-optimal if and only if

(i) $\lambda_{21} - \lambda_{11} + b_2 + \lambda_2 - 2r_2 < 0$ and

(ii) $$\rho = \frac{\lambda_{21} - \lambda_{11} + b_2 + \lambda_2 - 2r_2}{(r_1 - 2r_{11} + r_2 - b_2)^2 - (b_1 + b_2 - 1)(\lambda_{21} - \lambda_{11} + b_2 + \lambda_2 - 2r_2)},$$

where $r_{11} = 0.5\lambda_{11}(v-1)k_{21}^{-1}$.

Proof. If $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ then $\mathbf{X}'\mathbf{X} = (r_1 - \lambda_{21} + \lambda_{11} + r_2 + 2\rho_{22} - \lambda_2)\mathbf{I}_v +$ $+ (\lambda_{21} - \lambda_{11} + b_2 + \lambda_2 - 2r_2)\mathbf{1}_v\mathbf{1}_v'$. Hence, from Theorem 2 it follows that chemical balance weighing design is regular D-optimal if and only if conditions (i) and (ii) are simultaneously fulfilled. The condition $\mathbf{X}'\mathbf{1}_n = \mathbf{z}_p$ implies that $\mathbf{c}_j'\mathbf{X}'\mathbf{1}_n = m - 2u$ or $-(m-2u)$, $j = 1,2,...,p$, where $m - 2u = r_1 - 2r_{11} + r_2 - b_2$, $\mathbf{c}_j$ is jth column of the matrix $\mathbf{I}_p$. From $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p - \frac{\rho(m-2u)^2}{1+\rho(n-1)}(\mathbf{I}_p - \mathbf{1}_p\mathbf{1}_p')$ we obtain $\mathbf{c}_j'\mathbf{X}'\mathbf{X}\mathbf{c}_j = \frac{\rho(m-2u)^2}{1+\rho(n-1)}$ and

consequently $\mathbf{c}_j^{'}\mathbf{X}^{'}\mathbf{X}\mathbf{c}_{j^{'}} = \lambda_{21} - \lambda_{11} + b_2 + \lambda_2 - 2r_2$, $j \neq j^{'}$. Moreover, $\lambda_{21} - \lambda_{11} + b_2 + \lambda_2 - 2r_2 = \dfrac{\rho(r_1 - 2r_{11} + r_2 - b_2)^2}{1+\rho(b_1 + b_2 - 1)}$ and the condition (ii) is true. Under (i), the denominator (ii) is greater than zero, hence $\rho < 0$, i.e. $-(n-1)^{-1} < \rho < 0$.

Subsequently, we give the theorems presenting the parameters of the balanced bipartite weighing designs and the ternary balanced block designs. Based on these parameters we form the incidence matrices and next, the design matrices of the regular D-optimal chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$.

Theorem 3.2. If for a given ρ, the parameters of the ternary balanced block design are equal to $v = k_2 = 2s+1$, $b_2 = r_2 = 2s+u$, $\lambda_2 = 2s+u-1$, $\rho_{12} = u$, $\rho_{22} = s$ and the balanced bipartite weighing design are equal to

(i) $\rho = -2(5s^2 + 6s + 2u - 2)^{-1}$ and $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 3s$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$, $s = 2,4,7$,

(ii) $\rho = -(4s^2 + 3s + u - 1)^{-1}$ and $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 6s$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$, $s = 4,5,7$,

(iii) $\rho = -3(10s^2 + 9s + 3u - 3)^{-1}$ and $v = 2s+1$, $b_1 = s(2s+1)$, $r_1 = 8s$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$, $s = 4,5,7$,

$u = 1,2,...$, then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is as 1.1, is regular D-optimal.

Proof. It follows immediately that that the parameters given above satisfy the conditions (i) and (ii) of Theorem 3.1.

Theorem 3.3. If for a given ρ, the parameters of the ternary balanced block design are equal to $v = k_2 = 2s+1$, $b_2 = r_2 = 4s+u+1$, $\lambda_2 = 4s+u-1$, $\rho_{12} = u+1$, $\rho_{22} = 2s$ and the parameters of the balanced bipartite weighing design are equal to

(i) $\rho = -3\left(7s^2 + 15s + 3u\right)^{-1}$ and $v = 2s+1,\ b_1 = s(2s+1),\ r_1 = 3s,\ k_{11} = 1,$ $k_{21} = 2,\ \lambda_{11} = 2,\ \lambda_{21} = 1,\ s = 2,3,...$

(ii) $\rho = -3\left(10s^2 + 15s + 3u\right)^{-1}$ and $v = 2s+1,\ b_1 = s(2s+1),\ r_1 = 6s,$ $k_{11} = 2,\ k_{21} = 4,\ \lambda_{11} = 8,\ \lambda_{21} = 7,\ s = 3,4,...,$

(iii) $\rho = -\left(3s^2 + 5s + u\right)^{-1}$ and $v = 2s+1,\ b_1 = s(2s+1),\ r_1 = 8s,\ k_{11} = 3,$ $k_{21} = 5,\ \lambda_{11} = 15,\ \lambda_{21} = 13,\ s = 4,5,...,$

$u = 1,2,...$, then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is as 1.1, is regular D-optimal.

Proof. It is a simple matter to check that the parameters given above satisfy the conditions given in Theorem 3.1.

Theorem 3.4. If for a given ρ, the parameters of the ternary balanced block design are equal to $v = 2s+1,\ b_2 = u(2s+1),\ r_2 = u(2s-t+1),$ $k_2 = 2s-t+1,\ \lambda_2 = u(2s-2t+1),\ \rho_{12} = u\left(2s-t^2+1\right),\ \rho_{22} = 0.5ut(t-1)$ and the parameters of the balanced bipartite weighing design are equal to

(i) $\rho = -\left(3s^2 + u^2t^2 + s - 2sut + 2us + u - 1\right)^{-1}$ and $v = 2s+1,$ $b_1 = s(2s+1),\ r_1 = 3s,\ k_{11} = 1,\ k_{21} = 2,\ \lambda_{11} = 2,\ \lambda_{21} = 1,$

(ii) $\rho = -\left(6s^2 + u^2t^2 + s - 4sut + 2us + u - 1\right)^{-1}$ and $v = 2s+1,$ $b_1 = s(2s+1),\ r_1 = 6s,\ k_{11} = 2,\ k_{21} = 4,\ \lambda_{11} = 8,\ \lambda_{21} = 7,$

(iii) $\rho = -2\left(8s^2 + u^2t^2 + 2s - 4sut + 4us + 2u - 2\right)^{-1}$ and $v = 2s+1,$ $b_1 = s(2s+1),\ r_1 = 8s,\ k_{11} = 3,\ k_{21} = 5,\ \lambda_{11} = 15,\ \lambda_{21} = 13,$

where $u = 1,2,...$, for the cases (i) and (ii) $\begin{cases} t = 2 \quad \text{and} \quad s = 2,3,... \\ t = 3 \quad \text{and} \quad s = 5,6,... \\ t = 4 \quad \text{and} \quad s = 8,9,... \end{cases}$, for the case

(iii) $t = 2,3,4,\ s = 2t, 2t+1,...$, then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is as 1.1, is regular D-optimal.

Proof. In is evident, the parameters given above satisfy the conditions (i) and (ii) presented in Theorem 3.1.

Theorem 3.5. Let $v = 4s+1$. If for a given ρ, the parameters of the balanced bipartite weighing design and the ternary balanced block design are equal

(i) $\rho = -3\left(13s^2 + 27s + 3u\right)^{-1}$ and $b_1 = s(4s+1)$, $r_1 = 5s$, $k_{11} = 2$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 2$ and $b_2 = r_2 = 8s+u+1$, $k_2 = 4s+1$, $\lambda_2 = 8s+u-1$, $\rho_{12} = u+1$, $\rho_{22} = 4s$, $u = 1,2,\ldots$, $s = 2,3,\ldots$,

(ii) $\rho = -\left(5s^2 + u^2t^2 + s - 2sut + 4us + u - 1\right)^{-1}$ and $b_1 = s(4s+1)$, $r_1 = 5s$, $k_{11} = 2$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 2$ and $b_2 = u(4s+1)$, $r_2 = u(4s-t+1)$, $k_2 = 4s-t+1$, $\lambda_2 = u(4s-2t+1)$, $\rho_{12} = u\left(4s-t^2+1\right)$, $\rho_{22} = 0.5ut(t-1)$, $t = 2,3,4$, $s = t, t+1,\ldots$,

$u = 1,2,\ldots$, then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is as 1.1, is regular D-optimal.

Proof. Obviously, the parameters given in (i)-(ii) satisfy two conditions given in Theorem 3.1.

Theorem 3.6. If for a given ρ, the parameters of the balanced bipartite weighing design are equal to $v = 2s$, $b_1 = s(2s-1)$, $r_1 = 3(2s-1)$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and the parameters of the ternary balanced block design are equal to

(i) $\rho = -3\left(10s^2 + 5s + 3u - 8\right)^{-1}$ and $v = k_2 = 2s$, $b_2 = r_2 = 4s+u-2$, $\lambda_2 = 4s+u-4$, $\rho_{12} = u$, $\rho_{22} = 2s-1$, $s = 3,4,\ldots$,

(ii) $\rho = -\left(6s^2 + u^2t^2 - 5s + 2sut - 2ut + 2us\right)^{-1}$ and $v = 2s$, $b_2 = 2us$, $r_2 = u(2s-t)$, $k_2 = 2s-t$, $\lambda_2 = 2u(s-t)$, $\rho_{12} = u\left(2s-t^2\right)$, $\rho_{22} = 0.5ut(t-1)$, $\begin{cases} t = 2 & \text{and} \quad s = 3,4,\ldots \\ t = 3 & \text{and} \quad s = 5,6,\ldots \\ t = 4 & \text{and} \quad s = 9,10,\ldots \end{cases}$,

$u = 1,2,\ldots$, then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is as 1.1, is regular D-optimal.

Proof. Clearly, the parameters given above satisfy the conditions (i) and (ii) of Theorem 3.1.

Theorem 3.7. If for a given ρ, the parameters of the balanced bipartite weighing design are equal to $v = 6s$, $b_1 = 6s(6s-1)$, $r_1 = 3(6s-1)$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 4$, $\lambda_{21} = 2$ and the parameters of the ternary balanced block design are equal to

(i) $\rho = -4\left(180s^2 + 12s + 4u - 11\right)^{-1}$ and $v = k_2 = 6s$, $b_2 = r_2 = 12s + u - 2$, $\lambda_2 = 12s + u - 4$, $\rho_{12} = u$, $\rho_{22} = 6s - 1$, $s = 1,2,\ldots$,

(ii) $\rho = -2\left(108s^2 + u^2t^2 - 18s - 12sut + 2ut + 12us - 1\right)^{-1}$ and $v = 6s$, $b_2 = 6us$, $r_2 = u(6s-t)$, $k_2 = 6s - t$, $\lambda_2 = 2u(3s-t)$, $\rho_{12} = u\left(6s - t^2\right)$, $\rho_{22} = u$, $t = 2,3,4$, $s = t-1,\ t,\ t+1,\ldots$,

$u = 1,2,\ldots$, then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is as 1.1, is regular D-optimal.

Proof. It is easily seen that that the parameters given above satisfy the conditions (i) and (ii) of Theorem 3.1.

Theorem 3.8. If for a given ρ, the parameters of the balanced bipartite weighing design and the ternary balanced block design are equal to

(i) $\rho = -2\left(s^2 + 8s + 9\right)^{-1}$ and $v = 5$, $b_1 = 10$, $r_1 = 6$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ and $v = 5$, $b_2 = 5(s+1)$, $r_2 = 4(s+1)$, $k_2 = 4$, $\lambda_2 = 3s + 2$, $\rho_{12} = 4s$, $\rho_{22} = 2$, $s = 1,2,\ldots$,

(ii) $\rho = -\left(s^2 - 5s + 63\right)^{-1}$ and $v = 9$, $b_1 = 36$, $r_1 = 32$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 9$, $b_2 = 3(s+4)$, $r_2 = 2(s+4)$, $k_2 = 6$, $\lambda_2 = s + 5$, $\rho_{12} = 8$, $\rho_{22} = s$, $s = 1,2,\ldots$,

(iii) $\rho = -(s+27)^{-1}$ and $v = 9$, $b_1 = 18$, $r_1 = 10$, $k_{11} = 2$, $k_{21} = 3$, $\lambda_{11} = 3$, $\lambda_{21} = 2$ and $v = k_2 = 9$, $b_2 = r_2 = s+8$, $\lambda_2 = s+7$, $\rho_{12} = s$, $\rho_{22} = 4$, $s = 1,2,\ldots$,

(iv) $\rho = -1/101$ and $v = 11$, $b_1 = 55$, $r_1 = 40$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = k_2 = 11$, $b_2 = r_2 = 7$, $\lambda_2 = 4$, $\rho_{12} = 5$, $\rho_{22} = 1$,

(v) $\rho = -3/568$ and $v = 12$, $b_1 = 132$, $r_1 = 33$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 4$, $\lambda_{21} = 2$ and $v = 12$, $b_2 = 18$, $r_2 = 15$, $k_2 = 10$, $\lambda_2 = 11$, $\rho_{12} = 1$, $\rho_{22} = 7$,

(vi) $\rho = -0.5(2s^2 - 9s + 91)^{-1}$ and $v = 12$, $b_1 = 132$, $r_1 = 33$, $k_{11} = 1$, $k_{21} = 2$, $\lambda_{11} = 4$, $\lambda_{21} = 2$ and $v = 12$, $b_2 = 3(2s+5)$, $r_2 = 2(2s+5)$, $k_2 = 8$, $\lambda_2 = 2(s+3)$, $\rho_{12} = 6 - 2s$, $\rho_{22} = 3s + 2$, $s = 0,1,2$,

(vii) $\rho = -1/115$ and $v = 12$, $b_1 = 66$, $r_1 = 33$, $k_{11} = 2$, $k_{21} = 4$, $\lambda_{11} = 8$, $\lambda_{21} = 7$ and $v = 12$, $b_2 = 18$, $r_2 = 15$, $k_2 = 10$, $\lambda_2 = 11$, $\rho_{12} = 1$, $\rho_{22} = 7$,

(viii) $\rho = -0.33(3s^2 - 3s + 40)^{-1}$ and $v = 15$, $b_1 = 105$, $r_1 = 56$, $k_{11} = 3$, $k_{21} = 5$, $\lambda_{11} = 15$, $\lambda_{21} = 13$ and $v = 15$, $b_2 = 3(s+4)$, $r_2 = 2(s+4)$, $k_2 = 10$, $\lambda_2 = s+5$, $\rho_{12} = 6 - 2s$, $\rho_{22} = 2s + 1$, $s = 1,2$,

then the chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(-1,0,1)$ given by 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is as 1.1, is regular D-optimal.

Proof. It is easy to check that the parameters given above satisfy the conditions indicated in Theorem 3.1.

4. Discussion

Some constructions of the regular D-optimal chemical balance weighing designs with negative correlated errors based on the set of the incidence matrices of the balanced bipartite weighing designs were presented in Ceranka and Graczyk (2014c). Let us consider the class $\mathbf{\Phi}_{20\times 5}(-1,0,1)$. Here, the regular D-

optimal chemical balance weighing design was determined for $\rho = -1/55$ (Th. 2.3(xii), $s = 2$, $t = 1$), $\rho = -1/83$ (Th. 2.3(xiii), $s = 2$, $t = 1$). The regular D-optimal chemical balance weighing design was determined for $\rho = -1/20$ (Th. 6(i), $s = u = 1$), $\rho = -1/37$ (Th. 6(iii), $s = t = 1$, $u = 1$) in Ceranka and Graczyk (2015). Evidently, it is not possible to determine regular D-optimal chemical balance weighing design in any class $\mathbf{\Phi}_{n\times p}(-1,0,1)$ and for any value ρ. Hence, in this paper we give the matrix of regular D-optimal in class $\mathbf{\Phi}_{20\times 5}(-1,0,1)$ design for $\rho = -1/21$ (Th. 4(i), $s = 2$, $u = 6$), $\rho = -3/61$ (Th. 5(i), $s = 2$, $u = 1$), $\rho = -1/19$ (Th. 6(ii), $s = t = u = 2$), $\rho = -1/23$ (Th. 6(ii), $s = u = 2, t = 3$), $\rho = -1/18$ (Th. 10(i), $s = 1$). Similarly, for the class $\mathbf{\Phi}_{72\times 9}(-1,0,1)$, the constructions for $\rho = -3/469$ (Th. 2.3(vi), $s = 4$), $\rho = -1/199$ (Th. 2.3(x), $s = 4$, $t = 1$), $\rho = -1/215$ (Th. 2.3(xii), $s = 4$, $t = 1$), $\rho = -1/327$ (Th. 2.3(xiii), $s = 4$, $t = 1$), $\rho = -1/471$ (Th. 2.3(ix), $s = 4$) were presented in Ceranka and Graczyk (2014c). By contrast, in Ceranka and Graczyk (2015) the constructions for the cases $\rho = -1/72$ (Th. 6(i), $s = 2$, $u = 37$), $\rho = -1/169$ (Th. (iii), $s = 2$, $t = 1$, $u = 6$), $\rho = -1/135$ (Th. 11(iii), $s = 2$, $u = 40$), $\rho = -1/199$ (Th. 5, $s = 1$, $t = 4$, $u = 7$), $\rho = -1/471$ (Th. 11(ii), $u = 14$) were presented. The current study solves the existence problem in mentioned class for the cases $\rho = -1/79$ (Th. 4(i), $s = 4$, $u = 28$), $\rho = -1/103$ (Th. 4(ii), $s = 4$, $u = 28$), $\rho = -3/229$ (Th. 5(i), $s = 4$, $u = 19$), $\rho = -3/277$ (Th. 5(ii), $s = 4$, $u = 19$), $\rho = -1/87$ (Th. 5(iii), $s = 4$, $u = 19$), $\rho = -3/217$ (Th. 7(i), $s = 2$, $u = 37$), $\rho = -1/171$ (Th. 7(ii), $s = t = 2$, $u = 6$), $\rho = -1/73$ (Th. 10(iii), $s = 46$).

In the present paper, under fixed assumptions, we gave the solution of the problem how to determine the regular D-optimal chemical balance weighing

design in the classes in that we are not able to indicate optimal design by using of the methods know in the literature.

5. Example

Let us consider the experiment in that we determine unknown measurements of $p=5$ objects by used of $n=20$ measurements assuming that the correlation between measurement errors equals $\rho=-3/61$. We construct the matrix $\mathbf{X}\in\mathbf{\Phi}_{20\times5}(-1,0,1)$ according to the Theorem 5(i). Let $\mathbf{N}_1^*$ be the incidence matrix of the balanced bipartite weighing design with the parameters $v=5$, $b_1=10$, $r_1=6$, $k_{11}=1$, $k_{21}=2$, $\lambda_{11}=2$, $\lambda_{21}=1$ and let $\mathbf{N}_2$ be the incidence matrix of the ternary balanced block designs with the parameters $v=k_2=5$, $b_2=r_2=10$, $\lambda_2=8$, $\rho_{12}=2$, $\rho_{22}=4$ given in the forms

$$\mathbf{N}_1^*=\begin{bmatrix} 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_1 & 0 & 0 & 1_1 & 0 \\ 1_2 & 0 & 0 & 1_1 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 0 \\ 1_1 & 1_2 & 0 & 0 & 1_2 & 0 & 0 & 1_2 & 1_2 & 1_1 \\ 0 & 1_1 & 1_2 & 0 & 0 & 1_2 & 1_1 & 1_2 & 0 & 1_2 \\ 0 & 0 & 1_1 & 1_2 & 1_1 & 0 & 1_2 & 0 & 1_2 & 1_2 \end{bmatrix},$$

$$\mathbf{N}_2=\begin{bmatrix} 1 & 2 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 2 \\ 2 & 1 & 2 & 0 & 0 & 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 & 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 & 0 & 0 & 2 & 1 & 2 \\ 2 & 0 & 0 & 2 & 1 & 2 & 0 & 0 & 2 & 1 \end{bmatrix},$$

where 1_ς denotes the element belonging to the ς^{th} subblock, respectively, $\varsigma=1,2$.

According to the formula 3.1 we form the matrix $\mathbf{X}\in\mathbf{\Phi}_{20\times5}(-1,0,1)$ of the regular D-optimal chemical balance weighing design as

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & -1 \\ -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

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